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GRAVITY ANOMALIES AND THE FIGURE OF THE EARTH

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INTRODUCTION

The last few years have seen a remarkable activity in gravity observations in different parts of the globe, and with the advent of new instruments of improved patterns there is every reason to look forward to a rapid accumulation of further observational material. Side by side, there has been a considerable output of research on the theoretical aspects. The literature on the subject is however scattered about in different books and periodicals which are often inaccessible. Apart from this, it is so voluminous that it is possible only for a comparatively few people to study each paper critically. Some of the problems are still a subject of considerable difference of opinion among experts, and it appears to be pertinent to take stock of what has been done so far.

The purpose of this publication is to provide an introduction to the fundamental problems of higher gravity, explaining the lines of investigations developed in recent times, and the practical applications of the various formulæ. It is hoped that this paper will be useful to a wide circle of readers, including the experts.

Chapter I deals with the various gravity formulæ with brief proofs. The expansions of the various terms in the gravity formulæ have been carried out in power series mostly with a view to their application to the case of the earth. The classical part has been dealt with extensively by Helmert in his Höheren Geodäsie Vol. II, but the notation employed there is unfamiliar to English-reading people, and the treatment is often confusing. The modern developments are scattered about in a number of foreign periodicals some of which are not readily available.

The practical derivation of the empirical gravity formulæ is reviewed in Chap. II. In particular, the intricate problem of the derivation of the ellipticity of the equator is discussed.

Chapter III gives an account of Clairaut's, Darwin's and de Sitter's theory of the figure of the earth. Clairaut gives a differential equation between the ellipticities of the internal level surfaces of the earth and the distribution of density, assuming that there is hydrostatic equilibrium inside. Actually the material of the earth is nearly in hydrostatic equilibrium below the depth of compensation which is of the order of 50 km. Clairaut's equation is not very tractable to solution and has exercised the ingenuity of the early mathematicians, who had to make certain *ad hoc* assumptions about the variations of density in the earth's interior. These laws of density have now been definitely disproved and consequently these solutions have been purposely skipped over in this book as being of purely historical interest. A solution based on our modern concept of density distribution inside the earth as evidenced by seismological research is however included. The most reliable method of determining the ellipticity of the geoid is outlined.

Gravity anomalies founded on the various theories of compensation can be put to several uses, one of them being the determination of the inequalities of density in the crust. This is dealt with in Chapter IV. It is pointed out that the problem has not a unique solution even if no account is taken of the stresses in the earth's crust. Some useful formulæ have also been incorporated for obtaining the numerical estimates of the mass anomalies from the gravity profiles. The part played by gravity data in elucidating the tectonic folding in the various regions of the globe is also discussed. Gravity anomalies provide a direct measure of the excess or underload. Regions of very large positive anomalies, being areas of overload, should be expected to be continuously sinking, which is by no means always the case. A notable exception is that of the Island of Cyprus, which has risen in spite of being a region of large positive anomalies. Again, the upheaval of land in Fennoscandia does not bear a close correlation with the gravity anomalies.

Another use of the gravity anomalies is the determination of the form of the geoid, and the deflection of the plumb-line. The necessary formulæ and their practical applications are discussed in Chapter V.

Chapter VI deals with the question of the choice of a reference figure. The reference spheroids in vogue in geodetic work are so diverse that it is indispensable to have a clear conception of the conditions they have to satisfy before we can make a proper use of them. The problem of the linking up of gravity and deflection data is also elucidated.

For convenience, the diagrams have all been put together at the end of the book in a double page so that they can be opened clear of the text. A list of symbols is also given for easy reference.

Lastly, I would like to acknowledge my indebtedness to Mr. A. N. Ramanathan, M.A. for seeing the book through the press and for verifying some of the formulæ.

LIST OF SYMBOLS

θ = geocentric latitude

ϕ = geographical latitude

L = longitude reckoned positive east of Greenwich

L_0 = longitude of one end of the major axis of a triaxial ellipsoid

$Y_n, Z_n, S_n, u_n, v_n, H_n$ = Laplace's functions of order n

P_n = Legendre function of degree n

μ = sin θ (unless otherwise stated)

k = radius of a sphere having the same volume as a nearly spherical surface

a, b, c = principal semi-axes of a triaxial ellipsoid

M = total attracting mass of a body

$\epsilon = (a-c)/a$

$\eta = (a-b)/a$

ρ = volume density

σ = surface density

f = gravitational constant

g = gravity at a point

g_a, g_b, g_c = gravity at the extremities of the principal axes of a triaxial ellipsoid

G = mean value of gravity taken over a whole surface

G' = mean equatorial gravity on a triaxial ellipsoid

G_e = equatorial value of gravity on a spheroid

G_p = gravity at the extremity of the minor axis of a spheroid

γ_0 = normal value of gravity

ω = angular velocity of a rotating body

$m = \omega^2 a/G_e$

$m' = \omega^2 k/G$

$m'' = \omega^2 k/fM$

A, B, C = principal moments of inertia of a body

N = height of the compensated geoid above the reference spheroid

N_c = height of the natural geoid above the compensated geoid

$\Delta g = g - \gamma_0$ = conventional gravity anomaly

Δg_A = Free-air anomaly

Δg_B = Bouguer anomaly

Δg_C = Isostatic anomaly

Δg_{CH} = Hayford anomaly with respect to the Helmert's formula

Δg_{CI} = Hayford anomaly with respect to the International formula

τ = depth of compensation

η = meridional deflection (It has been used in this sense in chapters v and vi only and should not be confused with the equatorial ellipticity η)

ξ = prime vertical deflection

$0(\epsilon^2)$ = residual containing terms in ϵ^2 and higher orders. By the statement 'correct to $0(\epsilon^2)$' is implied that terms of order higher than ϵ^2 have been neglected.

In the text, by true or natural geoid is meant the geoid arrived at from deflections or gravity, when the actual topography is not interfered with. By compensated geoid is meant the geoid deduced on the basis of compensated topography. The difference between the compensated and natural geoids is, therefore, the deformation produced by the compensated topography.

CHAPTER I

THEORETICAL BASIS OF GRAVITY FORMULÆ

1. Definitions.—In the theory of the gravity field of the earth, the following surfaces are generally involved: true spheroid, level spheroid, triaxial ellipsoid and nearly spherical level surface. The term 'level spheroid' (*Niveausphäroid*) was coined by Helmert, and denotes a surface whose radius vector differs from that of a true spheroid by 10 or 15 feet (see para 6). It will be shown later (chapter III) that a homogeneous triaxial ellipsoid is not a possible form of equilibrium of a rotating fluid. If the earth were homogeneous, this surface would obviously be ruled out from any discussion of its gravity field. But this being not so, a triaxial ellipsoid plays an important role both in the theory of the figure of the earth as well as in the determination of its external field.

A nearly spherical surface is defined by an equation of the form

$$r=a\left\{1+\sum_{n=0}^{n} Y_n(\theta, L)\right\}, \qquad \dots \quad (1\cdot1)$$

where $Y_n(\theta, L)$ is a Laplace's function of order n, and θ, L are the geocentric latitude and longitude respectively.

$$\begin{aligned} Y_n(\theta, L) = A_n P_n(\mu) &+ (A_{n1}\cos L + B_{n1}\sin L)\, P_{n1}(\mu) \\ &+ (A_{n2}\cos 2L + B_{n2}\sin 2L)\, P_{n2}(\mu) \\ &+ \dots\dots + (A_{nn}\cos nL + B_{nn}\sin nL)\, P_{nn}(\mu), \end{aligned}$$

where $\mu = \sin\theta$ and $P_{nr}(\mu) = \cos^r\theta \dfrac{d^r}{d\mu^r} P_n(\mu)$. $P_n(\mu)$ is called a Legendre's function of degree n.

The interpretation of the various harmonic terms in equation (1·1) will be considered in chap. VI, para 2. The coefficient A_2 in the Laplace's function $Y_2(\theta, L)$ defines the mean meridional ellipticity of the surface, and is of the first order of small quantities. All the other A's are assumed to be of order A_2^2 or smaller.

The radius vector of the surface (1·1) differs from that of a sphere of radius a by $a\Sigma Y_n$. Its volume correct to terms of the second order is equal to that of a sphere of radius $a(1+Y_0)$, and the co-ordinates of its centre of gravity, assuming it to be a homogeneous body, are $\bar{z} = aA_1, \bar{y} = aA_{11}, \bar{x} = aB_{11}$, where A_1, A_{11}, B_{11} are constant coefficients in the expansion for $Y_1(\theta, L)$.

If then we choose the origin at the centre of gravity, we can write the equation (1·1) in the form

$$r = k\left(1+\sum_{n=2}^{n} Y_n\right), \qquad \dots \quad (1\cdot2)$$

where $k = a(1+Y_0)$ is the radius of a sphere of equal volume.

The case when there is only one Legendre harmonic present is very illustrative, as it lends itself to an easy geometric interpretation. $r = a\,(1+\epsilon P_n)$ is the equation of a surface differing from a sphere by n undulations of varying amplitudes, the maximum amplitude being $a\epsilon$.

2. Potential of a static homogeneous triaxial ellipsoid with special application to the case of the earth.— The expressions for the internal and external potentials of a homogeneous ellipsoid were given by Rodrigues* in 1815. The proofs are given in Routh's Statics, vol. II.

(*a*) *Internal potential.*—

Let the ellipsoid be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$ ... (1·3)

Let $a > b > c$, and let $\epsilon = \dfrac{a-c}{a}$ denote the meridional ellipticity of the section of the ellipsoid by the plane $y=0$, and $\eta = \dfrac{a-b}{a}$ the equatorial ellipticity. The potential† at an internal point (x, y, z) of the above ellipsoid of uniform density ρ is

$$U_i = \pi f abc\rho\,(A_{00} - A_{10}\,x^2 - A_{20}\,y^2 - A_{30}\,z^2), \qquad \dots \quad (1\cdot 4)$$

where f is the gravitational constant,

$$\left.\begin{aligned} \text{and } A_{00} &= \int_0^\infty \frac{ds}{\sqrt{\psi(s)}} \\ A_{10} &= \int_0^\infty \frac{ds}{(a^2+s)\sqrt{\psi(s)}} \\ A_{20} &= \int_0^\infty \frac{ds}{(b^2+s)\sqrt{\psi(s)}} \\ A_{30} &= \int_0^\infty \frac{ds}{(c^2+s)\sqrt{\psi(s)}} \\ \psi(s) &= (a^2+s)\,(b^2+s)\,(c^2+s). \end{aligned}\right\} \qquad \dots \quad (1\cdot 5)$$

For numerical work, the coefficients A_{00}, A_{10}, A_{20}, A_{30} can be evaluated by expressing them in terms of normal elliptic integrals. But for the case of the earth, ϵ, η are small, and it is more illustrative to expand the integrand in powers of ϵ and η.

Putting $\dfrac{a^2+s}{a^2} = v$, we have

$$a\,A_{00} = \int_1^\infty \frac{dv}{\sqrt{v\,(v-2\epsilon+\epsilon^2)\,(v-2\eta+\eta^2)}},$$

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* Correspondance sur l'Ecole Royale Polytechnique, vol. III.

† Routh, in Analytical Statics, vol. II, § 211, has deduced this from *a priori* considerations. In Clarke's Geodesy, p. 69, this expression is deduced by utilizing the lemma, that the potentials of confocal ellipsoids at an external point are in the ratio of their masses.

$$a^3 A_{10} = \int_1^{\infty} \frac{dv}{v\sqrt{v(v-2\epsilon+\epsilon^2)(v-2\eta+\eta^2)}},$$

$$a^3 A_{20} = \int_1^{\infty} \frac{dv}{(v-2\eta+\eta^2)\sqrt{v(v-2\epsilon+\epsilon^2)(v-2\eta+\eta^2)}},$$

$$a^3 A_{30} = \int_1^{\infty} \frac{dv}{(v-2\epsilon+\epsilon^2)\sqrt{v(v-2\epsilon+\epsilon^2)(v-2\eta+\eta^2)}}.$$

Expressing the integrand as a power series in ϵ, η and integrating, we have

$$\left.\begin{aligned} a A_{00} &= 2\left[1 + \frac{1}{3}(\epsilon+\eta) + \frac{2}{15}(\epsilon^2+\eta^2) + \frac{1}{5}\epsilon\eta\right] \\ a^3 A_{10} &= 2\left[\frac{1}{3} + \frac{1}{5}(\epsilon+\eta) + \frac{4}{35}(\epsilon^2+\eta^2) + \frac{1}{7}\epsilon\eta\right] \\ a^3 A_{20} &= 2\left[\frac{1}{3} + \frac{1}{5}(\epsilon+3\eta) + \frac{1}{35}(4\epsilon^2+27\eta^2) + \frac{3}{7}\epsilon\eta\right] \\ a^3 A_{30} &= 2\left[\frac{1}{3} + \frac{1}{5}(3\epsilon+\eta) + \frac{1}{35}(27\epsilon^2+4\eta^2) + \frac{3}{7}\epsilon\eta\right]. \end{aligned}\right\} \quad \ldots (1\cdot 6)$$

We will make use of these expressions in para 4, when we find the variation of gravity on the ellipsoid.

The accuracy of the above expressions $(1\cdot 6)$ can be checked by the fact, that when they are substituted in $(1\cdot 4)$, the resulting expression for U_i satisfies Poisson's equation.

$$\begin{aligned} \nabla^2 U_i &= -2\pi f abc\,\rho\,(A_{10}+A_{20}+A_{30}) \\ &= -\frac{4\pi f bc}{a^2}\,\rho\left[1+(\epsilon+\eta)+(\epsilon^2+\eta^2+\epsilon\eta)\right] \\ &= -4\pi f\rho\left[1-(\epsilon+\eta)+\epsilon\eta\right]\left[1+(\epsilon+\eta)+(\epsilon^2+\eta^2+\epsilon\eta)\right] \\ &= -4\pi f\rho. \end{aligned}$$

(*b*) *External potential* *.—The potential of an ellipsoid at an external point (x, y, z) is

$$U_e = \pi f\rho\, abc\,(A'_{00} - A'_{10}x^2 - A'_{20}y^2 - A'_{30}z^2), \quad \ldots \quad (1\cdot 7)$$

$$\left.\begin{aligned} \text{where } A'_{00} &= \int_u^{\infty} \frac{ds}{\sqrt{\psi(s)}} \\ A'_{10} &= \int_u^{\infty} \frac{ds}{(a^2+s)\sqrt{\psi(s)}} \\ A'_{20} &= \int_u^{\infty} \frac{ds}{(b^2+s)\sqrt{\psi(s)}} \\ A'_{30} &= \int_u^{\infty} \frac{ds}{(c^2+s)\sqrt{\psi(s)}} \\ \psi(s) &= (a^2+s)(b^2+s)(c^2+s). \end{aligned}\right\} \quad \ldots \quad (1\cdot 8)$$

* Routh, Analytical Statics, vol. II, § 225.

The parameter u is given by the positive root of the cubic equation

$$x^2(b^2+u)\ (c^2+u)+y^2(c^2+u)\ (a^2+u)+z^2(a^2+u)\ (b^2+u) - (a^2+u)\ (b^2+u)\ (c^2+u) = 0. \qquad \ldots \quad (1\cdot 9)$$

Comparing equation $(1\cdot 7)$ with $(1\cdot 4)$, we see that the forms of the internal and external potentials are exactly similar. There is however the fundamental difference, that while in the former the coefficients A are constant quantities, in the latter they are functions of a variable u depending on the position of the point. For $u = 0$ corresponding to a point on the surface of the ellipsoid, the two expressions become identical.

The external potential may also be written in the form

$$U_e = \pi f \, \frac{abc\rho}{a'b'c'} \left\{ B_{00} - B_{10}x^2 - B_{20}y^2 - B_{30}z^2 \right\},$$

where a', b', c' are the semi-axes of the confocal ellipsoid through the external point (x, y, z), and B_{00}, B_{10}, B_{20}, B_{30} are the same functions of a', b', c' as A_{00}, A_{10}, A_{20}, A_{30} in equation $(1\cdot 4)$ are of (a, b, c).

3. Internal and external level surfaces of a homogeneous ellipsoid.—

From equation $(1\cdot 4)$, we see that the internal equipotentials are

$$A_{10}x^2 + A_{20}y^2 + A_{30}z^2 = A_{00}. \qquad \ldots \quad (1\cdot 10)$$

The equatorial ellipticity of these surfaces is

$$\eta_1 = \frac{\dfrac{1}{\sqrt{A_{10}}} - \dfrac{1}{\sqrt{A_{20}}}}{\dfrac{1}{\sqrt{A_{10}}}} = \frac{\sqrt{A_{20}} - \sqrt{A_{10}}}{\sqrt{A_{20}}}.$$

The meridional ellipticity is

$$\epsilon_1 = \frac{\dfrac{1}{\sqrt{A_{10}}} - \dfrac{1}{\sqrt{A_{30}}}}{\dfrac{1}{\sqrt{A_{10}}}} = \frac{\sqrt{A_{30}} - \sqrt{A_{10}}}{\sqrt{A_{30}}}.$$

Now
$$A_{10} \doteq \frac{2}{3a^3}\left[1 + \tfrac{3}{5}(\epsilon + \eta)\right],$$

$$A_{20} \doteq \frac{2}{3a^3}\left[1 + \tfrac{3}{5}(\epsilon + 3\eta)\right],$$

$$A_{30} \doteq \frac{2}{3a^3}\left[1 + \tfrac{3}{5}(3\epsilon + \eta)\right].$$

Hence $\eta_1 = \frac{3}{5}\eta$ and $\epsilon_1 = \frac{3}{5}\epsilon$.

All the internal level surfaces have therefore the same ellipticity, which is 40 % less than that of the attracting ellipsoid.

Hence the internal level surfaces of an attracting ellipsoid are similar and similarly situated ellipsoids, but they are not confocal. Also these surfaces are more spherical than the bounding surface.

The properties of the external level surfaces are difficult to elucidate. From (1·7), we see that their equation is

$$A'_{10}x^2 + A'_{20}y^2 + A'_{30}z^2 = A'_{00}. \qquad \dots \quad (1\cdot 11)$$

The coefficients A'_{10}, A'_{20}, etc., are functions of a parameter u defined by the complicated equation (1·9).

4. Gravity on a homogeneous ellipsoid and spheroid.—For obtaining the force of attraction on a homogeneous triaxial ellipsoid, we will make use of the formula (1·4) for the internal potential. The external potential (1·7) is unsuitable, as the coefficients of the various terms in it are infinite integrals with lower limit u, which can not easily be evaluated.

The components of gravity at a point (x, y, z) are given by

$$\left.\begin{aligned} g_x &= -\frac{\delta U_i}{\delta x} = +\ 2\pi f abc\,\rho\, A_{10}x \\ g_y &= -\frac{\delta U_i}{\delta y} = +\ 2\pi f abc\,\rho\, A_{20}y \\ g_z &= -\frac{\delta U_i}{\delta z} = +\ 2\pi f abc\,\rho\, A_{30}z. \end{aligned}\right\} \qquad \dots \quad (1\cdot 12)$$

The resultant gravity is

$$g = 2\pi f abc\,\rho\,\sqrt{A_{10}^2 x^2 + A_{20}^2 y^2 + A_{30}^2 z^2}. \qquad \dots \quad (1\cdot 13)$$

If ϕ denotes the geographical latitude, the co-ordinates of a point on the ellipsoid can be written as

$$x = \frac{a\cos\phi\cos L}{Q},\ y = \frac{a\,(1-e_1^2)\cos\phi\,\sin L}{Q},\ z = \frac{a\,(1-e_2^2)\sin\phi}{Q},$$

where $Q = \sqrt{\cos^2\phi\cos^2 L + (1-e_1^2)\cos^2\phi\sin^2 L + (1-e_2^2)\sin^2\phi}$,

$$\text{and} \qquad e_1^2 = \frac{a^2-b^2}{a^2},\ e_2^2 = \frac{a^2-c^2}{a^2}.$$

$$\text{Hence } g = \frac{2\pi f abc\,\rho}{Q}\Big\{a^2A_{10}^2\cos^2\phi\cos^2 L + a^2A_{20}^2(1-e_1^2)^2 \times$$

$$\cos^2\phi\sin^2 L + a^2A_{30}^2(1-e_2^2)^2\sin^2\phi\Big\}^{\frac{1}{2}}. \dots \quad (1\cdot 14)$$

Substituting values of A_{10}, A_{20}, A_{30} from (1·6), and retaining terms up to order ϵ^2 only, we have

$$g = \frac{4\pi\rho f bc}{3a}\Big[1 + \epsilon\Big(\frac{3}{5} + \frac{1}{5}\sin^2\phi\Big) + \epsilon^2\Big(\frac{12}{35} - \frac{73}{350}\sin^2\phi$$

$$+ \frac{19}{50}\sin^4\phi\Big) + \eta\Big(\frac{3}{5} + \frac{1}{10}\cos^2\phi - \frac{1}{10}\cos^2\phi\cos 2L\Big)\Big], \quad \dots (1\cdot 15)$$

$$= \frac{fM}{a^2}\Big(1 + D' + A'\sin^2\phi - B'\sin^2 2\phi + C'\cos^2\phi\cos 2L\Big),$$

where M denotes the mass of the attracting matter, and

$$\left.\begin{aligned} D' &= \frac{3}{5}\epsilon + \frac{12}{35}\epsilon^2 + \frac{7}{10}\eta \\ A' &= \frac{1}{5}\epsilon + \frac{6}{35}\epsilon^2 - \frac{1}{10}\eta \\ B' &= \frac{19}{200}\epsilon^2 \\ C' &= -\frac{1}{10}\eta. \end{aligned}\right\}$$

The magnitude of the neglected terms is of the order $G\epsilon^3$ which only amounts to about $\frac{1}{30}$ mgal if G is taken as 1000 gals.

The values of gravity at the extremities (A, B, C) of the three axes are

$$g_a = \frac{fM}{a^2}\left(1 + \frac{3}{5}\epsilon + \frac{12}{35}\epsilon^2 + \frac{3}{5}\eta\right), \text{ corresponding to } \phi = 0,\ L = 0.$$

$$g_b = \frac{fM}{a^2}\left(1 + \frac{3}{5}\epsilon + \frac{12}{35}\epsilon^2 + \frac{4}{5}\eta\right), \quad \text{,,} \quad \text{,,}\ \phi = 0, L = 90^\circ.$$

$$g_c = \frac{fM}{a^2}\left(1 + \frac{4}{5}\epsilon + \frac{18}{35}\epsilon^2 + \frac{3}{5}\eta\right), \quad \text{,,} \quad \text{,,}\ \phi = 90^\circ.$$

Obviously $g_c > g_a$ and $g_b > g_a$.

Also $g_c - g_b = \dfrac{fM}{a^2}\left(\dfrac{1}{5}\epsilon + \dfrac{6}{35}\epsilon^2 - \dfrac{1}{5}\eta\right) > 0.$

Hence $g_c > g_b > g_a$, which is what is expected.

But this relation does not hold for a rotating ellipsoid, as we shall see later.

Again, the components of gravity vector at any point (x, y, z) are

$$\left.\begin{aligned} g_x &= -\frac{\delta U_i}{\delta x} = 2\kappa A_{10}\, x \\ g_y &= -\frac{\delta U_i}{\delta y} = 2\kappa A_{20}\, y \\ g_z &= -\frac{\delta U_i}{\delta z} = 2\kappa A_{30}\, z, \end{aligned}\right\}$$

where A_{10}, A_{20}, A_{30} are given by (1·6), and $\kappa = \pi f\rho\, abc$.

It is obvious that the resultant gravity vector on a *homogeneous* triaxial ellipsoid does not coincide with the normal to the surface, because g_x, g_y, g_z are not proportional to the direction cosines of the normal at the point. Hence a *homogeneous* triaxial ellipsoid cannot be a level surface of the masses within it. But there is nothing to prevent a triaxial ellipsoid from being a surface of equilibrium of a system of masses. It is well-known that the level surfaces of a thin shell bounded by concentric and similar ellipsoids are confocal ellipsoids. If our triaxial ellipsoid is a member of this family, then so far as the external field is concerned, the masses inside it are equivalent to the thin shell bounded by similar ellipsoids. This is, of course, provided that all the masses are inside the ellipsoid. Hence the external level surfaces of the masses, which make the boundary of an ellipsoid a level surface, are ellipsoids confocal with the given one.

The case of the spheroid (a, ϵ) is deduced from the above by putting $\eta = 0$. It will be readily seen that even on a homogeneous spheroid, gravity vector is not along the normal.

The radius of a sphere of equal volume is

$$k = a\left[1 - \frac{1}{3}(\epsilon + \eta) - \frac{1}{9}\epsilon^2\right] = 0\cdot998{,}864\ a.$$

Gravity at a point on this sphere is

$$g_o = \frac{fM}{k^2} = 1\cdot002{,}276\frac{fM}{a^2}.$$

The above equations enable us to compare the values of gravity on a triaxial ellipsoid and a sphere of equal volume.

For a nearly spherical ellipsoid, the following method can be used to give an approximate value of gravity.

Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ be an ellipsoid with a small ellipticity.

The radius k of a sphere of equal volume is given by $k^3 = abc$.

Put $a = k\,(1 + \epsilon_1)$, $b = k\,(1 + \epsilon_2)$, $c = k\,(1 + \epsilon_3)$,

then

$$(1 + \epsilon_1)\,(1 + \epsilon_2)\,(1 + \epsilon_3) = 1,$$

or neglecting higher powers then the first,

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0.$$

The ellipsoid can now be written in the form

$$\Sigma\ \frac{x^2}{k^2\,(1 + \epsilon_1)^2} = 1,$$

or

$$x^2 + y^2 + z^2 = k^2 + 2\,(\epsilon_1 x^2 + \epsilon_2 y^2 + \epsilon_3 z^2),$$

or

$$r^2 = k^2 + 2\,(\epsilon_1 x^2 + \epsilon_2 y^2 + \epsilon_3 z^2).$$

Hence

$$r = k + \frac{1}{k}\,\Sigma\epsilon_1 x^2.$$

The potential at any point may be regarded as due to a sphere of volume density ρ, and a coating of surface density $\dfrac{\rho}{k}\Sigma\epsilon_1 x^2$ on this sphere. The potentials at internal and external points are

$$\left.\begin{aligned} V_i &= \frac{2}{3}\pi\rho\,(3k^2 - r^2) + \frac{4}{5}\pi\rho\,(\epsilon_1 x^2 + \epsilon_2 y^2 + \epsilon_3 z^2) \\ V_e &= \frac{4}{3}\pi\rho\frac{k^3}{r} + \frac{4}{5}\pi\rho\,(\epsilon_1 x^2 + \epsilon_2 y^2 + \epsilon_3 z^2)\left(\frac{k}{r}\right)^5. \end{aligned}\right\}$$

Gravity at an internal point of the sphere is

$$g_i = \sqrt{\Sigma \left(\frac{\delta V_i}{\delta x}\right)^2},$$

$$= \pi\rho\sqrt{x^2\left(\frac{8}{5}\epsilon_1 - \frac{4}{3}\right)^2 + y^2\left(\frac{8}{5}\epsilon_2 - \frac{4}{3}\right)^2 + z^2\left(\frac{8}{5}\epsilon_3 - \frac{4}{3}\right)^2}. \quad \ldots \quad (1\cdot 16)$$

This may be written in the same form as (1·13), namely

$$g = 2\pi f\rho abc\sqrt{B_{10}{}^2x^2 + B_{20}{}^2y^2 + B_{30}{}^2z^2},$$

where

$$a^3B_{10} = 2\left[\frac{1}{3} + \frac{1}{5}(\epsilon + \eta) + \frac{1}{9}\epsilon^2\right],$$

$$a^3B_{20} = 2\left[\frac{1}{3} + \frac{1}{5}(\epsilon + 3\eta) + \frac{1}{9}\epsilon^2\right],$$

$$a^3B_{30} = 2\left[\frac{1}{3} + \frac{1}{5}(3\epsilon + \eta) + \frac{29}{45}\epsilon^2\right].$$

Comparison with equation (1·6) reveals, that the above coefficients are identical with the corresponding A's up to first order terms.

Substituting $x = \dfrac{a\cos\phi\cos L}{Q}$, $y = \dfrac{b^2\cos\phi\sin L}{aQ}$, $z = \dfrac{c^2\sin\phi}{aQ}$ in (1·16), we obtain the value of gravity on the ellipsoid to be

$$g_i = \frac{fM}{a^2}\left[1 + \left(\frac{3}{5}\epsilon + \frac{7}{10}\eta + \frac{1}{3}\epsilon^2\right) + \sin^2\phi\left(\frac{1}{5}\epsilon - \frac{1}{10}\eta - \frac{1}{5}\epsilon^2\right) - \frac{1}{10}\eta\cos^2\phi\cos 2L - \frac{19}{200}\epsilon^2\sin^2 2\phi\right].$$

The ellipsoid is partly internal and partly external to the sphere, and strictly speaking, this formula applies only to those portions of the ellipsoid which are within the sphere. It is also approximate, since the three-dimensional mass $\frac{\rho}{k}\Sigma\epsilon_1 x^2$ above the sphere is replaced by a coating.

Comparing it with the rigorous formula (1·15), we see that the error of this approximate formula is

$$\Delta g = \frac{fM}{a^2}\left(\frac{1}{105}\epsilon^2 + \frac{13}{35}\epsilon^2\sin^2\phi\right).$$

The maximum error is at the pole, and amounts to 4 mgals. The minimum error is about 0·1 mgal at the equator.

The corresponding formula for gravity derived from the expression V_e for the external potential is

$$g_e = \frac{fM}{a^2}\left\{1 + \left(\frac{3}{5}\epsilon + \frac{7}{10}\eta\right) + \sin^2\phi\left(\frac{1}{5}\epsilon - \frac{1}{10}\eta - \frac{6}{5}\epsilon^2\right) - \frac{1}{10}\eta\cos^2\phi\cos 2L + \frac{89}{50}\epsilon^2\sin^2 2\phi\right\}.$$

As mentioned before, this method involves a condensation of matter of thickness of about 4 miles at the equator and 8 miles at the pole, and is necessarily approximate.

The application of this method to the case of a nearly spherical surface $r = a\ (1 + \sum_{n=1} \epsilon_n Y_n)$ will be readily understood after a perusal of chap. v. At this stage it need only be mentioned that if we take as a reference surface the spheroid $r = a\ (1 + \epsilon_1 Y_1 + \epsilon_2 Y_2)$, having the same ellipticity as the original surface, then the difference in the values of gravity on the two surfaces can be obtained very accurately by considering the effect of a coating of skin density $a\rho \sum_{n=3} \epsilon_n Y_n$ on a sphere of radius a.

5. Rotating bodies.—We have so far considered only static homogeneous bodies. The application of these formulæ to the case of the actual earth is obviously very limited, because we know that the earth is rotating, and also that it is heterogeneous.

The expression for the external potential of a homogeneous rotating spheroidal earth is $W = U + \frac{1}{2}\omega^2 r^2 \cos^2\theta$, where *

$$U = \frac{fM}{r}\left\{1 + \frac{a^2e^2}{10r^2}(1 - 3\sin^2\theta) + \frac{a^4e^4}{280r^4}(105\sin^4\theta - 90\sin^2\theta + 9) + \dots\right\},$$

e being the eccentricity. From this we can derive the expression for gravity on the rotating spheroid.

In the general case, however, the body is not homogeneous, and its internal constitution is not known. We will now show that the gravity field can still be determined provided the surface is an equipotential. The extra condition that the surface is an equipotential is required to compensate for our lack of knowledge of the internal mass distribution.

6. Gravity formulæ for rotating bodies.—For a static homogeneous ellipsoid, we started with an expression for the internal potential. This is not possible for a heterogeneous body whose internal law of density is not known. The problem of finding gravity on such a surface is soluble for certain types of rotating bodies, and that only when the outer surface of the body is an equipotential. There are two main directions into which the body of research into gravity formulæ may be branched. One is the classical method of Stokes, Helmert and Darwin, and the other is the modern work of Pizetti, Somigliana, Cassini and other continental writers.

Stokes'† solution is embodied in his famous paper "On the variation of gravity at the surface of the earth" and is applicable to a nearly spherical surface. The polar equation of such a surface is

$$r = k\ (1 + \sum_{1}^{\infty} u_n), \qquad \dots \quad (1 \cdot 17)$$

where k denotes the radius of a sphere of equal volume.

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* Routh's Statics, vol. II, § 303.

† Mathematical and Physical Papers, **2**, 1883, 131–71.

The potential at an external point due to matter within this surface is

$$W = \Sigma \frac{fY_n}{r^{n+1}} + \frac{1}{2}\,\omega^2 r^2 \cos^2\theta.$$

The surface being an equipotential, we will have $W = W_0$ at all points on it. This condition gives

$$W = fY_0\Big(\frac{1}{r} + \frac{ku_1}{r^2} + \frac{k^2u_2}{r^3} + \ldots\Big) - \frac{\omega^2k^5}{2r^3}\Big(\frac{1}{3} - \sin^2\theta\Big), \quad \ldots(1\cdot18)$$

where $fY_0 = kW_0 - \frac{1}{3}\omega^2k^3$; Y_0 is the mass of the matter inside the surface.

If dn, dr denote elements of length along the normal to the surface and radius vector of the sphere of equal volume respectively, then gravity on the surface is

$$g = -\frac{\delta W}{\delta n} \doteq -\frac{\delta W}{\delta r}$$

$$= G\left[1 - \frac{5}{2}m'\Big(\frac{1}{3} - \sin^2\theta\Big) + \sum_2^{\infty}(n-1)\,u_n\right], \quad \ldots\ (1\cdot19)$$

where

$$\left.\begin{aligned} G &= \frac{fY_0}{k^2} - \frac{2}{3}\,\omega^2 k \\ \text{and} \qquad m' &= \frac{\omega^2 k}{G}. \end{aligned}\right\}$$

G denotes the mean value of gravity over the whole surface. Knowing m', g and G, we can get k and u's. If then, we are given g at all points of a level surface having no masses external to it, the parameters defining the level surface are known with one reservation. All the u's can be determined in equation (1·17) except u_1, which determines the co-ordinates of the centre of gravity of the level surface. The gravity values therefore enable us to determine the ellipticity of the level surface, but not its orientation.

The converse problem is, "Given the form of the level surface, gravity on it is known except for one constant G or m', which must be obtained by some other method". We shall discuss this more fully in chapters v and vi when we are considering the reference surfaces for gravity work.

The formula (1·19) has a direct application to the case of the spheroid and the triaxial ellipsoid. The equation of an oblate spheroid correct to the first order in ellipticity ϵ is

$$r_s = k\left[1 - \epsilon\Big(\sin^2\theta - \frac{1}{3}\Big)\right].$$

Gravity on it is therefore

$$g_s = G\left[1 + \Big(\frac{5}{2}m' - \epsilon\Big)\Big(\sin^2\theta - \frac{1}{3}\Big)\right]. \qquad \ldots\ (1\cdot20)$$

The polar equation of a triaxial ellipsoid, the mean ellipticity of whose meridians is ϵ_0 and the ellipticity of whose equator is η, may be written as

$$r=k\left[1+\epsilon_0\left(\frac{1}{3}-\sin^2\theta\right)+\frac{1}{2}\eta\cos^2\theta\cos 2\left(L-L_0\right)\right], \quad \dots \quad (1\cdot21)$$

where L_0 is the longitude of the extremity of the major axis. By comparison with (1·17), we have

$$u_1=0,\ u_2=\epsilon_0\left(\frac{1}{3}-\sin^2\theta\right)+\frac{1}{2}\eta\cos^2\theta\cos 2\left(L-L_0\right).$$

Hence $g=G\left\{1+\left(\epsilon_0-\frac{5}{2}m'\right)\left(\frac{1}{3}-\sin^2\theta\right)+\frac{1}{2}\eta\cos^2\theta\cos 2\left(L-L_0\right)\right\}$,

$$\dots \quad (1\cdot22)$$

where $m'=\frac{\omega^2 k}{G}$ and $G=\frac{W_0}{k}-\omega^2 k$, W_0 denoting the value of the potential on the equipotential surface.

Caution however is required in utilising the above formulæ based on Stokes' paper, as they are derived from first order considerations only. Helmert* and Darwin† realized that they were inadequate to satisfy the practical requirements of geodesy, and extended the gravity formulæ to second order terms in ellipticity, their methods being practically identical. They proceeded from the following expression for the potential at an external point P due to attracting matter within a surface.

$$U_P=\iiint\frac{f\,dm}{\sqrt{R^2+r^2-2Rr\cos\zeta}}=\sum_{n=0}^{n}\frac{fY_n}{r^{n+1}}\ \text{for } r>R, \quad \dots \quad (1\cdot23)$$

where $Y_n=\iiint\limits_{R=0} R^n P_n\,dm.$

r, R denote respectively the distances of the point P and an element of attracting mass dm from the centre of mass, and ζ is the angle between the directions of R and r.

Choosing the origin at the centre of mass of the attracting system, and the axes of inertia as the axes of co-ordinates, the first three terms of (1·23) can easily be evaluated. We have

$Y_0=M$, the total attracting mass,

$Y_1=0$,

$$Y_2=\iiint R^2P_2\,dm=\frac{3}{2}\left(\frac{A+B}{2}-C\right)\left(\sin^2\theta-\frac{1}{3}\right)+\frac{3}{4}(B-A)\cos^2\theta\cos 2L,$$

where A, B, C are the principal moments of inertia of the body. Hence, the complete expression for the potential W is

$$W=\frac{fM}{r}\left\{1-\frac{3}{2}\frac{K}{r^2}\left(\sin^2\theta-\frac{1}{3}\right)+\frac{3}{4}\frac{B-A}{Mr^2}\cos^2\theta\cos 2L\right\}+\frac{fY_3}{r^4}+\frac{fY_4}{r^5}$$

$$+\dots\dots+\frac{1}{2}\omega^2r^2\cos^2\theta, \quad \dots \quad (1\cdot24)$$

where $K=\frac{1}{M}\left(-\frac{A+B}{2}+C\right)$.

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* Höheren Geodäsie, **2**, 1884, 50–130.

† Scientific Papers, **3**, 1910, 78.

The essence of this method is, that although the constitution of the body is unknown, the leading terms of the potential can be evaluated in terms of certain constants of the body. Equation (1·24) represents the potential of any rotating body. In deriving it, we have not made use of the condition that the boundary of the body is an equipotential surface. The form of the surface and its internal constitution not being known, one cannot proceed much further with the evaluation of gravity on it. Helmert, however, used equation (1·24) to give a clue to the potentials of the level surfaces of the earth, which he designated by 'level spheroids'. The forms of these spheroids and the values of gravity on them have to be connected, so that given one we can find the other. This cannot be done for the actual earth, because for connecting g and r one extra condition is needed, namely, that W is constant on the surface. This condition does not hold for the earth. Helmert* assumed the potential of the level spheroids to be

$$U = \frac{fM}{r}\left[1 + \frac{K}{2r^2}(1-3\sin^2\theta) - \frac{3}{4r^2}\left(\frac{A-B}{M}\right)\cos^2\theta\cos 2L + \frac{1}{2}\frac{\omega^2 r^3\cos^2\theta}{fM}\right]. \quad \ldots \quad (1\cdot25)$$

In this, the quantities Y_3, Y_4, etc., have been neglected, and A, B are no longer the exact moments of inertia of a level spheroid. Each level spheroid is characterised by the value of U on its surface.

Since we are now aiming at accuracy up to second order of small quantities, we have to take

$$g^2 = U_1^2 + U_2^2,$$

where
$$U_1 = -\frac{\delta U}{\delta r}, \quad U_2 = -\frac{\delta U}{r\delta\theta}.$$

Gravity at an external point so deduced is

$$g = \frac{fM}{r^2}\left\{1 + \frac{3K}{2r^2}(1-3\sin^2\theta) + \frac{9(B-A)}{4Mr^2}\cos^2\theta\cos 2L - \frac{\omega^2 r^3}{fM}\cos^2\theta\right\}.$$

Helmert does not develop this equation further. Attention may however be directed to the fact that if we assume the form of the equipotential to be

$$r = k\left\{1 + \epsilon_0\left(\frac{1}{3} - \sin^2\theta\right) + \frac{1}{2}\eta\cos^2\theta\cos 2(L-L_0)\right\},$$

the formula for gravity (to the first order in ϵ_0 and η) reduces to

$$g = G\left\{1 + \left(\epsilon_0 - \frac{5}{2}m'\right)\left(\frac{1}{3} - \sin^2\theta\right) + \frac{1}{2}\eta\cos^2\theta\cos 2(L-L_0)\right\},$$

which is identical with equation (1·22). If U_0 denotes the value of the potential on the level spheroid, the relations between the

* Höheren Geodäsie, 2, 1884, 72.

various constants are

$$\left.\begin{aligned} U_0 &= \frac{fM}{k}\left(1+\frac{1}{3}m'\right)\\ G &= \frac{fM}{k^2}\left(1-\frac{2}{3}m'\right)\\ m' &= \frac{\omega^2 k}{G}\\ \frac{K}{k^2} &= \frac{2}{3}\left(\epsilon-\frac{1}{2}m'\right)\\ \frac{B-A}{Mk^2} &= \frac{2}{3}\eta. \end{aligned}\right\}$$

It is worth pointing out that the expression (1·25) for the potential must be supplemented by a term $\frac{\beta P_4}{r^5}$ before it can be used for deriving the gravity formula correct to $0(\epsilon^2)$ on the surface

$$r = k\left\{1+\epsilon_0\left(\frac{1}{3}-\sin^2\theta\right)+\frac{1}{2}\eta\cos^2\theta\cos 2(L-L_0)\right\}.$$

Helmert* next took as an approximation to W,

$$U = \frac{fM}{r}\left\{1+\frac{K}{2r^2}(1-3\sin^2\theta)+\frac{\omega^2 r^3}{2fM}\cos^2\theta \right.$$
$$\left. +\frac{D}{r^4}\left(\sin^4\theta-\frac{6}{7}\sin^2\theta+\frac{3}{35}\right)\right\}, \quad \ldots \quad (1\cdot26)$$

and proceeding as before found the equation of the level spheroid $U = W_0$ to be

$$r = a\left\{1-\left(\epsilon-2\epsilon^2+\frac{5}{2}\epsilon m+\delta\right)\sin^2\theta-\left(2\epsilon^2-\frac{5}{2}\epsilon m-\delta\right)\sin^4\theta\right\} \quad \ldots \quad (1\cdot27)$$

and gravity on it to be

$$g = G_e\left\{1+\left(\frac{5}{2}m-\epsilon+6\epsilon^2-\frac{1}{2}\epsilon m-\frac{19}{7}\delta\right)\sin^2\theta-(7\epsilon^2-3\delta)\sin^4\theta\right\}, \quad \ldots \quad (1\cdot28)$$

where G_e denotes the mean value of gravity on the equator.

$$\left.\begin{aligned} G_e &= \frac{fM}{a^2}\left\{1+\epsilon-\frac{3}{2}m-\epsilon^2-\frac{1}{2}\epsilon m+\frac{9}{4}m^2+\frac{4}{7}\delta\right\}\\ m &= \frac{\omega^2 a}{G_e}\\ \delta &= \frac{D}{a^4}\\ \frac{3K}{2a^2} &= \epsilon-\frac{1}{2}m-\epsilon^2+\frac{1}{2}\epsilon m+\frac{3}{4}m^2+\frac{1}{7}\delta\\ a &= \frac{fM}{W_0}\left(1+\frac{1}{3}\epsilon+\frac{1}{3}m-\frac{1}{3}\epsilon^2+\frac{2}{3}\epsilon m-\frac{1}{2}m^2+\frac{2}{15}\delta\right). \end{aligned}\right\} \quad (1\cdot29)$$

In terms of geographical latitude ϕ, equation (1·28) may be written as

$$g = G_e(1+A'\sin^2\phi-B'\sin^2 2\phi), \quad \ldots \quad (1\cdot30)$$

* Höheren Geodäsie, **2**, 1884, 89.

where $$\left.\begin{aligned} A' &= -\epsilon + \frac{5}{2}m - \epsilon^2 - \frac{1}{2}\epsilon m + \frac{2}{7}\delta \\ 4B' &= 3\delta - 7\epsilon^2 + 4\epsilon A'. \end{aligned}\right\} \qquad \ldots \quad (1\cdot31)$$

The mean value of gravity on surface $(1\cdot27)$ is

$$G = \frac{U_0^2}{fM}\left(1 - \frac{4}{3}m - \frac{8}{45}\epsilon^2 + \frac{25}{9}m^2\right). \qquad \ldots \quad (1\cdot32)$$

Since this is independent of δ, we see that the mean value of gravity on the family of surfaces $(1\cdot27)$ is the same for different values of δ.

If we take $\delta = \frac{7}{2}\epsilon^2 - \frac{5}{2}\epsilon m$, the surface $(1\cdot27)$ becomes a true spheroid. If this spheroid is an equipotential of its internal masses, gravity on it is

$$g = G_e\left\{1 + \left(\frac{5}{2}m - \epsilon - \frac{7}{2}\epsilon^2 + \frac{44}{7}m\epsilon\right)\sin^2\theta + \left(\frac{7}{2}\epsilon^2 - \frac{15}{2}m\epsilon\right)\sin^4\theta\right\}, \qquad \ldots \quad (1\cdot33)$$

where $$G_e = \frac{fM}{a^2}\left(1 + \epsilon - \frac{3}{2}m + \epsilon^2 - \frac{27}{14}\epsilon m + \frac{9}{4}m^2\right). \qquad \ldots \quad (1\cdot34)$$

In terms of geographical latitude ϕ, gravity on a true spheroid is

$$g = G_e\left[1 + A'\sin^2\phi - B'\sin^2 2\phi\right], \qquad \ldots \quad (1\cdot35)$$

where $$\left.\begin{aligned} A' &= \frac{5}{2}m - \epsilon - \frac{17}{14}\epsilon m \\ B' &= \frac{1}{8}\epsilon(5m - \epsilon) \\ m &= \frac{\omega^2 a}{G_e}. \end{aligned}\right\} \qquad \ldots \quad (1\cdot36)$$

Darwin starts with the level spheroid

$$r = a\left(1 - \epsilon\sin^2\theta - \frac{3}{2}\epsilon^2\sin^2\theta\cos^2\theta + \chi\sin^2\theta\cos^2\theta\right), \qquad \ldots \quad (1\cdot37)$$

which becomes identical with $(1\cdot27)$ if χ is taken equal to $\frac{7}{2}\epsilon^2 - \frac{5}{2}m\epsilon - \delta$.

Equation $(1\cdot37)$ may be written as

$$\begin{aligned} r &= a\left[\left(1 + \frac{2}{15}\chi - \frac{1}{3}\epsilon - \frac{1}{5}\epsilon^2\right) + P_2\left(\frac{2}{21}\chi - \frac{2}{3}\epsilon - \frac{1}{7}\epsilon^2\right) - P_4\left(\frac{8}{35}\chi - \frac{12}{35}\epsilon^2\right)\right] \\ &= k\left[1 + P_2\left(\frac{2}{21}\chi - \frac{2}{3}\epsilon - \frac{23}{63}\epsilon^2\right) - P_4\left(\frac{8}{35}\chi - \frac{12}{35}\epsilon^2\right)\right], \end{aligned} \qquad \ldots \quad (1\cdot38)$$

where $$\begin{aligned} k &= a\left(1 + \frac{2}{15}\chi - \frac{1}{3}\epsilon - \frac{1}{5}\epsilon^2\right) \\ &= \frac{fM}{W_0}\left\{1 + \frac{1}{3}m - \frac{8}{45}\epsilon^2 + \frac{2}{9}\epsilon m - \frac{1}{2}m^2\right\}. \end{aligned}$$

Darwin's formula for gravity on this surface is

$$g = G\Big[1+\Big(\frac{5}{3}m-\frac{2}{3}\epsilon+\frac{64}{63}m\epsilon-\frac{5}{9}\epsilon^2-\frac{25}{18}m^2+\frac{2}{21}\chi\Big)P_2 - \Big(\frac{12}{7}m\epsilon-\frac{4}{5}\epsilon^2+\frac{24}{35}\chi\Big)P_4\Big]\cdots \qquad (1\cdot 39)$$

G denotes the mean value of gravity on $(1\cdot 38)$, and is given by the expression

$$\left.\begin{aligned} G &= \frac{fM}{k^2}\Big(1-\frac{2}{3}m+\frac{4}{9}m\epsilon+m^2-\frac{8}{15}\epsilon^2\Big)\\ &= G_e\Big(1+\frac{5}{6}m-\frac{1}{3}\epsilon+\frac{25}{42}m\epsilon-\frac{7}{15}\epsilon^2+\frac{32}{105}\chi\Big)\\ &= \frac{fM}{k^2}\Big(1-\frac{2}{3}m'+\frac{4}{9}m'^2-\frac{8}{15}\epsilon^2+\frac{4}{9}\epsilon m'\Big), \end{aligned}\right\} \quad \cdots \quad (1\cdot 40)$$

where $m' = \dfrac{\omega^2 k}{G}$.

The equatorial value of gravity is

$$\left.\begin{aligned} G_e &= \frac{fM}{a^2}\Big(1-\frac{3}{2}m+\epsilon-\frac{27}{14}m\epsilon+\frac{9}{4}m^2+\epsilon^2-\frac{4}{7}\chi\Big)\\ &= \frac{fM}{k^2}\Big(1-\frac{3}{2}m+\frac{1}{3}\epsilon-\frac{13}{14}m\epsilon+\frac{9}{4}m^2+\frac{2}{45}\epsilon^2-\frac{32}{105}\chi\Big). \end{aligned}\right\} \quad (1\cdot 41)$$

It should be noted that while G_e depends on χ, G does not. In other words, the mean values of gravity on a spheroid and a surface whose radius vector differs from it by $a\chi \sin^2\theta \cos^2\theta$ are the same.

The parameters δ and χ define the radial separation of a level spheroid from a true spheroid. It is to be observed, that they do not affect either the mean value of gravity on the surface, or the mean radius. The surface $(1\cdot 37)$ differs from a true spheroid having the same axes by $a\chi \sin^2\theta\cos^2\theta$. We shall see in the next chapter that χ is of the order 200×10^{-8}. The level spheroid approximating to the geoid of the earth can therefore differ at the most by ten or fifteen feet from a true spheroid.

An extension of the above result is, that if the geoid is

$$r = k\Big(1-\frac{2}{3}\epsilon P_2+\sum_2^\infty u_n\Big), \qquad \cdots \quad (1\cdot 42)$$

gravity on it is

$$g = G\,[\,1+\alpha P_2+\beta P_4+\sum_2^\infty (n-1)\,u_n\,], \qquad \cdots \quad (1\cdot 43)$$

$$\left.\begin{aligned} \text{where } \alpha &= \frac{1}{3}(5m'-2\epsilon)+\frac{64}{63}m'\epsilon-\frac{4}{21}\epsilon^2\\ \beta &= -\frac{4}{35}(15\,\epsilon m'+2\epsilon^2)\\ G &= \frac{fM}{k^2}\Big(1-\frac{2}{3}m'+\frac{4}{9}m'^2-\frac{8}{15}\epsilon^2+\frac{4}{9}\epsilon m'\Big)\\ m' &= \frac{\omega^2 k}{G}. \end{aligned}\right\} \quad \cdots \quad (1\cdot 44)$$

This is a more precise form of equation (1·19) which was based on first order considerations only.

It is interesting to show that this formula is correct to terms of order ϵ^2 by considering its application to an ellipsoid. The equation of a triaxial ellipsoid is

$$r=a\left(1-\frac{1}{3}\epsilon-\frac{1}{3}\eta-\frac{1}{5}\epsilon^2\right)\left[1+P_2\left(-\frac{2}{3}\epsilon-\frac{23}{63}\epsilon^2+\frac{1}{3}\eta\right)\right.$$
$$\left.+\frac{12}{35}\epsilon^2P_4+\frac{1}{2}\eta\cos^2\theta\cos 2L\right]$$

$$=k\left[1-\frac{2}{3}\epsilon P_2+\left(\frac{1}{3}\eta-\frac{23}{63}\epsilon^2\right)P_2+\frac{12}{35}\epsilon^2P_4+\frac{1}{2}\eta\cos^2\theta\cos 2L\right],$$

where ϵ denotes the ellipticity of the meridional section through the xz plane, and η the equatorial ellipticity.

Comparing with (1·42), we have

$$\left.\begin{aligned}u_2&=\left(\frac{1}{3}\eta-\frac{23}{63}\epsilon^2\right)P_2+\frac{1}{2}\eta\cos^2\theta\cos 2L\\ u_4&=\frac{12}{35}\epsilon^2P_4,\ u_3=u_5=\text{etc.}=0.\end{aligned}\right\}$$

Hence, by (1·43),

$$g=G\left[1+\alpha P_2+\beta P_4+\left(\frac{1}{3}\eta-\frac{23}{63}\epsilon^2\right)P_2+\frac{36}{35}\epsilon^2P_4\right.$$
$$\left.+\frac{1}{2}\eta\cos^2\theta\cos 2L\right]$$

$$=G\left[1+\frac{1}{2}\left(3\sin^2\theta-1\right)\left(\alpha+\frac{1}{3}\eta-\frac{23}{63}\epsilon^2\right)\right.$$
$$\left.+\frac{1}{8}\left(35\sin^4\theta-30\sin^2\theta+3\right)\left(\beta+\frac{36}{35}\epsilon^2\right)+\frac{1}{2}\eta\cos^2\theta\cos 2L\right]$$

$$=G\left[1-\frac{1}{2}\alpha-\frac{1}{6}\eta+\frac{179}{315}\epsilon^2+\frac{3}{8}\beta\right.$$
$$+\sin^2\theta\left(\frac{3}{2}\alpha+\frac{1}{2}\eta-\frac{185}{42}\epsilon^2-\frac{15}{4}\beta\right)+\sin^4\theta\left(\frac{35}{8}\beta+\frac{9}{2}\epsilon^2\right)$$
$$\left.+\frac{1}{2}\eta\cos^2\theta\cos 2L\right].$$

The constants α, β contain a variable m' defined by $m'=\dfrac{\omega^2 k}{G}$. It can be easily shown that it is connected with the variable $m=\dfrac{\omega^2 a}{g_a}$ by the relation $m'=m\left(1-\dfrac{5}{6}m\right)$; g_a denotes the value of gravity at the point $\theta=0$, $L=0$. Putting in the values of α, β in terms of m, we have

$$g = G\left[\left(1 - \frac{5}{6}m + \frac{1}{3}\epsilon + \frac{25}{36}m^2 + \frac{26}{45}\epsilon^2 - \frac{1}{6}\eta - \frac{145}{126}m\epsilon\right) + \sin^2\theta\left(-\epsilon + \frac{5}{2}m + \frac{1}{2}\eta + \frac{167}{21}m\epsilon - \frac{25}{12}m^2 - \frac{23}{6}\epsilon^2\right) + \sin^4\theta\left(-\frac{15}{2}m\epsilon + \frac{7}{2}\epsilon^2\right) + \frac{1}{2}\eta\cos^2\theta\cos 2L\right].$$

To get this expression in terms of geographical latitude ϕ, we put $\theta = \phi - \epsilon\sin 2\phi$. Then

$$g = G\left[\left(1 - \frac{5}{6}m + \frac{1}{3}\epsilon + \frac{25}{36}m^2 + \frac{26}{45}\epsilon^2 - \frac{1}{6}\eta - \frac{145}{126}m\epsilon\right) + \sin^2\phi\left(-\epsilon + \frac{5}{2}m + \frac{1}{2}\eta + \frac{19}{42}m\epsilon - \frac{25}{12}m^2 - \frac{1}{3}\epsilon^2\right) + \sin^2 2\phi\left(\frac{1}{8}\epsilon^2 - \frac{5}{8}m\epsilon\right) + \frac{1}{2}\eta\cos^2\phi\cos 2L\right]$$

$$= G'\left[1 + A'\sin^2\phi - B'\sin^2 2\phi + C'\cos^2\phi\cos 2L\right],$$

where

$$G' = G\left(1 - \frac{5}{6}m + \frac{1}{3}\epsilon + \frac{25}{36}m^2 + \frac{26}{45}\epsilon^2 - \frac{1}{6}\eta - \frac{145}{126}m\epsilon\right)$$

and
$$G = \frac{fM}{k^2}\left(1 - \frac{2}{3}m + m^2 - \frac{8}{15}\epsilon^2 + \frac{4}{9}\epsilon m\right) = \frac{fM}{a^2}\left(1 + \frac{2}{3}\epsilon + \frac{2}{3}\eta - \frac{2}{3}m + \frac{1}{5}\epsilon^2 + m^2\right).$$

Hence
$$\left.\begin{aligned} G' &= \frac{fM}{a^2}\left[1 + \epsilon + \frac{1}{2}\eta - \frac{3}{2}m + \epsilon^2 + \frac{9}{4}m^2 - \frac{27}{14}m\epsilon\right]. \\ \text{Also}\quad A' &= -\epsilon + \frac{5}{2}m + \frac{1}{2}\eta - \frac{17}{14}m\epsilon \\ B' &= \frac{5}{8}m\epsilon - \frac{1}{8}\epsilon^2 \\ C' &= \frac{1}{2}\eta. \end{aligned}\right\}$$

This agrees precisely with our later formula (1·73) obtained by Pizetti's method, which is correct up to terms of order ϵ^2.

To complete this discussion, we will give the gravity formulæ on the two surfaces

$$r = a\,(1 - \epsilon\sin^2\theta) \qquad \ldots \quad (1\cdot 45)$$

$$\text{and}\quad r = k\left(1 - \frac{2}{3}\epsilon P_2\right). \qquad \ldots \quad (1\cdot 46)$$

The radii vectores of these two surfaces differ by terms of $O(a\epsilon^2)$, i.e. by 200 feet or so. The terms in ϵ^2 in the expressions for gravity on these two surfaces will naturally be different.

On the surface (1·45), gravity is

$$
\begin{aligned}
g &= G_e\,(1+\lambda\sin^2\theta+\mu\sin^2\theta\cos^2\theta)\\
&= G_e\left[\left(1+\frac{1}{3}\lambda+\frac{2}{15}\mu\right)+P_2\left(\frac{2}{3}\lambda+\frac{2}{21}\mu\right)-\frac{8}{35}\mu P_4\right]\\
&= G\,(1+\alpha P_2+\beta P_4)\\
&= G_e\,(1+A'\sin^2\phi-B'\sin^2 2\phi), \qquad \ldots \qquad (1\cdot47)
\end{aligned}
$$

where

$$
\left.
\begin{aligned}
\lambda &= \frac{5}{2}m-\epsilon-\frac{17}{14}m\epsilon-\frac{3}{7}\epsilon^2\\
\mu &= \frac{15}{2}m\epsilon+\epsilon^2\\
G_e &= \frac{fM}{a^2}\left(1-\frac{3}{2}m+\epsilon-\frac{27}{14}m\epsilon+\frac{9}{4}m^2+\frac{1}{7}\epsilon^2\right)\\
G &= \frac{fM}{a^2}\left(1-\frac{2}{3}m+\frac{2}{3}\epsilon+m^2-\frac{1}{5}\epsilon^2\right)\\
\alpha &= \frac{5}{3}m-\frac{2}{3}\epsilon+\frac{64}{63}m\epsilon-\frac{26}{63}\epsilon^2-\frac{25}{18}m^2\\
\beta &= -\frac{12}{7}m\epsilon-\frac{8}{35}\epsilon^2\\
A' &= \frac{5}{2}m-\epsilon-\frac{17}{14}m\epsilon-\frac{3}{7}\epsilon^2\\
B' &= \frac{5}{8}m\epsilon-\frac{5}{4}\epsilon^2\\
m &= \frac{\omega^2 a}{G_e}=\frac{\omega^2a^3}{fM}\left(1+\frac{3}{2}\frac{\omega^2a^3}{fM}-\epsilon\right).
\end{aligned}
\right\} \quad (1\cdot48)
$$

On surface (1·46),

$$
\begin{aligned}
g &= G_e\,(1+\lambda\sin^2\theta+\mu\sin^2\theta\cos^2\theta) \qquad \ldots \qquad (1\cdot49)\\
&= G\,(1+\alpha P_2+\beta P_4),
\end{aligned}
$$

where

$$
\left.
\begin{aligned}
\lambda &= \frac{5}{2}m-\epsilon-\frac{17}{14}m\epsilon-\frac{2}{21}\epsilon^2\\
\mu &= \frac{15}{2}\epsilon m+\epsilon^2\\
G_e &= \frac{fM}{k^2}\left(1-\frac{3}{2}m+\frac{1}{3}\epsilon+\frac{9}{4}m^2-\frac{13}{14}\epsilon m-\frac{11}{21}\epsilon^2\right)\\
G &= \frac{fM}{k^2}\left(1-\frac{2}{3}m'+\frac{4}{9}m'^2-\frac{8}{15}\epsilon^2+\frac{4}{9}\epsilon m'\right)\\
\alpha &= \frac{5}{3}m'-\frac{2}{3}\epsilon+\frac{64}{63}\epsilon m'-\frac{4}{21}\epsilon^2\\
\beta &= -\frac{12}{7}m'\epsilon-\frac{8}{35}\epsilon^2,\ \text{and}\ m'=\frac{\omega^2k}{G}.
\end{aligned}
\right\} \ldots \quad (1\cdot50)
$$

We will now consider Pizetti's * and Somigliana's † treatment.

* Principi della teoria meccanica della fijura d'equilibro dei planetti, Pisa, 1913.
† R. Accademia Delle Scienze, Torino, 1934.

Their method leads to elegant formulæ for gravity on an equipotential sphere, spheroid and triaxial ellipsoid. These formulæ do not involve power series. For numerical work, however, it is convenient to develop expansions in terms of ϵ, η and neglect terms of small order.

The external potential of any rotating body may be written as

$$W=\left(\frac{fM}{m_0}+\frac{1}{2}\omega^2\frac{m_1}{m_0}\right)V_0-\frac{1}{2}\omega^2V_1+\frac{1}{2}\omega^2(x^2+y^2), \quad \dots \quad (1\cdot51)$$

where V_0 and V_1 are two functions such that

$V_0=1$ and $V_1=(x^2+y^2)$ on the surface,

and

$$\underset{R\to\infty}{\text{Lt.}}\; RV_0=m_0, \text{ and } \underset{R\to\infty}{\text{Lt.}}\; RV_1=m_1.$$

For an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the appropriate expression for the external potential is

$$W=\tfrac{1}{2}fM\,V_0(u)-\frac{1}{2}\omega^2\left\{B_1V_{11}(u)+B_2V_{22}(u)\right\}+\frac{1}{2}\omega^2(x^2+y^2), \quad (1\cdot52)$$

where u denotes the parameter of the confocal ellipsoid which passes through the point (x, y, z) at which the potential is required, and is given by the positive root of the equation $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$.

B_1, B_2 are two constants which are determined from the condition that W is constant on the ellipsoid.

The functions V are given by the following expressions:

$$\left.\begin{aligned}
V_0(u)&=\int_u^\infty\frac{ds}{\sqrt{\psi(s)}}\\
V_{11}(u)&=\int_u^\infty\left(\frac{3x^2}{a^2+s}+\frac{y^2}{b^2+s}+\frac{z^2}{c^2+s}-1\right)\frac{ds}{(a^2+s)\sqrt{\psi(s)}}\\
V_{22}(u)&=\int_u^\infty\left(\frac{x^2}{a^2+s}+\frac{3y^2}{b^2+s}+\frac{z^2}{c^2+s}-1\right)\frac{ds}{(b^2+s)\sqrt{\psi(s)}},\\
&\text{where } \psi(s)=(a^2+s)(b^2+s)(c^2+s).
\end{aligned}\right\} \quad (1\cdot53)$$

Let

$$\left.\begin{aligned}
A'_{11}&=\int_u^\infty\frac{ds}{(a^2+s)^2\sqrt{\psi(s)}}, & A'_{10}&=\int_u^\infty\frac{ds}{(a^2+s)\sqrt{\psi(s)}}\\
A'_{12}&=\int_u^\infty\frac{ds}{(a^2+s)(b^2+s)\sqrt{\psi(s)}}, & A'_{20}&=\int_u^\infty\frac{ds}{(b^2+s)\sqrt{\psi(s)}}\\
A'_{22}&=\int_u^\infty\frac{ds}{(b^2+s)^2\sqrt{\psi(s)}}, & &\text{etc.}
\end{aligned}\right\} \quad (1\cdot54)$$

Then

$$\left.\begin{aligned}
V_{11}(u)&=3x^2A'_{11}+y^2A'_{12}+z^2A'_{13}-A'_{10}\\
V_{22}(u)&=x^2A'_{12}+3y^2A'_{22}+z^2A'_{23}-A'_{20}.
\end{aligned}\right\} \quad \dots \quad (1\cdot55)$$

Substitute these values of V_{11}, V_{22} in equation (1·52) and write

$$z^2 = (c^2+u) - \frac{c^2+u}{a^2+u}x^2 - \frac{c^2+u}{b^2+u}y^2.$$

The condition that W is constant for $u=0$ gives

$$\left.\begin{aligned}(3A_{11}a^2 - A_{13}c^2)B_1 + (A_{12}a^2 - A_{23}c^2)B_2 = a^2\\ (A_{12}b^2 - A_{13}c^2)B_1 + (3A_{22}b^2 - A_{23}c^2)B_2 = b^2.\end{aligned}\right\}\ldots \quad (1\cdot56)$$

In these equations A_{11}, A_{12}, etc., are infinite integrals defined by (1·54) with the lower limit of integration $u=0$.

If g_x denotes the component of gravity along the x axis, we have

$$g_x = -\frac{\delta W}{\delta x} = -\frac{1}{2}fM\frac{\delta V_0}{\delta x} + \frac{1}{2}\omega^2\left\{B_1\frac{\delta V_{11}}{\delta x} + B_2\frac{\delta V_{22}}{\delta x}\right\} - \omega^2 x. \quad (1\cdot57)$$

From equation (1·55), we have

$$\frac{\delta V_{11}}{\delta x} = 6xA'_{11} - \frac{1}{(a^2+u)\sqrt{\psi(u)}}\left\{\frac{3x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} - 1\right\}\frac{\delta u}{\delta x}$$

$$= 6xA'_{11} - \frac{2x^2}{(a^2+u)^2\sqrt{\psi(u)}}\frac{\delta u}{\delta x},$$

$$\frac{\delta V_{22}}{\delta x} = 2xA'_{12} - \frac{2y^2}{(b^2+u)^2\sqrt{\psi(u)}}\frac{\delta u}{\delta x}.$$

By (1·53),

$$\frac{\delta V_0}{\delta x} = \frac{\delta}{\delta x}\int_u^\infty \frac{ds}{\sqrt{\psi(s)}} = -\frac{1}{\sqrt{\psi(u)}}\frac{\delta u}{\delta x}.$$

Also since

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1,$$

we have $\dfrac{\delta u}{\delta x} = \dfrac{2x}{a^2+u}\left\{\dfrac{x^2}{(a^2+u)^2} + \dfrac{y^2}{(b^2+u)^2} + \dfrac{z^2}{(c^2+u)^2}\right\}^{-1}.$

Substituting these values in (1·57), we have

$$\left.\begin{aligned} g_x &= x\left[\frac{fM-\omega^2K(u)}{h^2(u)\sqrt{\psi(u)}}\cdot\frac{1}{a^2+u} + \omega^2P(u)\right]\cdot\\ \text{Similarly } g_y &= y\left[\frac{fM-\omega^2K(u)}{h^2(u)\sqrt{\psi(u)}}\cdot\frac{1}{b^2+u} + \omega^2Q(u)\right],\\ g_z &= z\left[\frac{fM-\omega^2K(u)}{h^2(u)\sqrt{\psi(u)}}\cdot\frac{1}{c^2+u} + \omega^2R(u)\right],\end{aligned}\right\} \quad (1\cdot58)$$

$$\left.\begin{aligned}\text{where } h^2(u) &= \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\\ K(u) &= 2\left[\frac{B_1x^2}{(a^2+u)^2} + \frac{B_2y^2}{(b^2+u)^2}\right]\\ P &= 3A'_{11}B_1 + A'_{12}B_2 - 1\\ Q &= A'_{12}B_1 + 3A'_{22}B_2 - 1\\ R &= A'_{13}B_1 + A'_{23}B_2.\end{aligned}\right\}\ldots \quad (1\cdot59)$$

Putting $u=0$, the components of gravity on our ellipsoid at a point (x, y, z) are

$$\left.\begin{aligned} g_x &= \left[\frac{fM-\omega^2 K(0)}{abc\,h^2(0)} + \omega^2 E\right]\frac{x}{a^2} \\ g_y &= \left[\frac{fM-\omega^2 K(0)}{abc\,h^2(0)} + \omega^2 E\right]\frac{y}{b^2} \\ g_z &= \left[\frac{fM-\omega^2 K(0)}{abc\,h^2(0)} + \omega^2 E\right]\frac{z}{c^2}, \end{aligned}\right\} \quad \ldots \quad (1\cdot 60)$$

where $E = a^2 P_0 = b^2 Q_0 = c^2 R_0$ and P_0, Q_0, R_0 are the values of P, Q, R for $u=0$.

The surface being equipotential, the resulting direction of gravity is along the normal, whose direction cosines are

$$\frac{x}{a^2 h(0)},\ \frac{y}{b^2 h(0)},\ \frac{z}{c^2 h(0)}.$$

Hence $$g = \frac{x g_x}{a^2 h(0)} + \frac{y g_y}{b^2 h(0)} + \frac{z g_z}{c^2 h(0)} = \frac{fM-\omega^2 K(0)}{abc\,h(0)} + \omega^2 E\,h(0). \quad (1\cdot 61)$$

If g_a, g_b, g_c denote the values of gravity at the extremities of the three axes, we have from $(1\cdot 61)$

$$\left.\begin{aligned} \frac{g_a}{a} &= \frac{1}{abc}\left(fM - \frac{2B_1\omega^2}{a^2}\right) + \frac{\omega^2 E}{a^2} \\ \frac{g_b}{b} &= \frac{1}{abc}\left(fM - \frac{2B_2\omega^2}{b^2}\right) + \frac{\omega^2 E}{b^2} \\ \frac{g_c}{c} &= \frac{1}{abc}(fM) + \frac{\omega^2 E}{c^2}. \end{aligned}\right\} \quad \ldots \quad (1\cdot 62)$$

Hence $$\frac{g_a}{a}\cdot\frac{x^2}{a^2} + \frac{g_b}{b}\cdot\frac{y^2}{b^2} + \frac{g_c}{c}\cdot\frac{z^2}{c^2} = \frac{1}{abc}fM - \frac{\omega^2}{abc}K(0) + \omega^2 E h^2(0) = g \times h(0),$$

or $$g = \frac{1}{h(0)}\left(\frac{g_a}{a}\cdot\frac{x^2}{a^2} + \frac{g_b}{b}\cdot\frac{y^2}{b^2} + \frac{g_c}{c}\cdot\frac{z^2}{c^2}\right)$$

$$= \frac{(a g_a \cos^2 L + b g_b \sin^2 L)\cos^2\phi + c g_c \sin^2\phi}{\sqrt{(a^2\cos^2 L + b^2 \sin^2 L)\cos^2\phi + c^2\sin^2\phi}}$$

$$= g_a\,\frac{1 + p\sin^2\phi + q\sin^2 L\cos^2\phi}{\sqrt{1 - e_2^2\sin^2\phi - e_1^2\sin^2 L\cos^2\phi}}, \quad \ldots \quad (1\cdot 63)$$

where $$\left.\begin{aligned} p &= \frac{c g_c - a g_a}{a g_a}, \quad q = \frac{b g_b - a g_a}{a g_a}, \\ e_1^2 &= \frac{a^2-b^2}{a^2} = 2\eta - \eta^2, \\ e_2^2 &= \frac{a^2-c^2}{a^2} = 2\epsilon - \epsilon^2. \end{aligned}\right\} \quad \ldots \quad (1\cdot 64)$$

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Expanding the right hand side of (1·63) in an infinite series, and retaining only terms up to order ϵ^2, Somigliana* has obtained the expression

$$g = g_a \left[1 + \frac{1}{2} (e_2^2 + 2p) \sin^2 \phi + \frac{1}{2} (e_1^2 + 2q) \cos^2 \phi \sin^2 L + \frac{1}{8} e_2^2 (3e_2^2 + 4p) \sin^4 \phi \right]. \quad \ldots \quad (1 \cdot 65)$$

The above summarizes Pizetti's and Somigliana's treatment for a triaxial ellipsoid. As these formulæ stand, the various constants are expressed in terms of infinite integrals, and it is neither easy to know the orders of magnitude of the various terms, nor possible to compare the formulæ with the older ones based on classical treatment. It is of interest to find expressions for these constants in terms of ϵ, η which we know to be small for the case of the earth. The results are as follows.

The values of A_{11}, A_{12}, &c. occuring in equation (1·56) correct to the second powers of ϵ and η are

$$\left.\begin{aligned}
A_{11} &= \frac{2}{5a^5}\left(1 + \frac{5}{7}\epsilon + \frac{5}{7}\eta + \frac{10}{21}\epsilon^2 + \frac{10}{21}\eta^2 + \frac{5}{9}\epsilon\eta \right) \\
A_{12} &= \frac{2}{5a^5}\left(1 + \frac{5}{7}\epsilon + \frac{15}{7}\eta + \frac{10}{21}\epsilon^2 + \frac{65}{21}\eta^2 + \frac{5}{3}\epsilon\eta \right) \\
A_{13} &= \frac{2}{5a^5}\left(1 + \frac{15}{7}\epsilon + \frac{5}{7}\eta + \frac{65}{21}\epsilon^2 + \frac{10}{21}\eta^2 + \frac{5}{3}\epsilon\eta \right) \\
A_{22} &= \frac{2}{5a^5}\left(1 + \frac{5}{7}\epsilon + \frac{25}{7}\eta + \frac{10}{21}\epsilon^2 + \frac{500}{63}\eta^2 + \frac{25}{9}\epsilon\eta \right) \\
A_{23} &= \frac{2}{5a^5}\left(1 + \frac{15}{7}\epsilon + \frac{15}{7}\eta + \frac{65}{21}\epsilon^2 + \frac{65}{21}\eta^2 + 5\epsilon\eta \right).
\end{aligned}\right\} \quad (1 \cdot 66)$$

Substituting these values in equation (1·56), we have

$$\left.\begin{aligned}
B_1 &= \frac{5}{4}a^5\left(1 - \frac{9}{7}\epsilon - \frac{5}{7}\eta + \frac{25}{49}\epsilon^2 + \frac{5}{147}\eta^2 + \frac{191}{147}\epsilon\eta \right) \\
B_2 &= \frac{5}{4}a^5\left(1 - \frac{9}{7}\epsilon - 3\eta + \frac{25}{49}\epsilon^2 + \frac{565}{147}\eta^2 + \frac{415}{147}\epsilon\eta \right).
\end{aligned}\right\} \quad (1 \cdot 67)$$

The coefficients P_0, Q_0, R_0 in equation (1·60) are

$$\left.\begin{aligned}
P_0 &= \left(1 - \frac{8}{7}\epsilon - \frac{3}{7}\eta + \frac{20}{147}\epsilon^2 + \frac{25}{98}\eta^2 + \frac{64}{147}\epsilon\eta \right) \\
Q_0 &= \left(1 - \frac{8}{7}\epsilon + \frac{11}{7}\eta + \frac{20}{147}\epsilon^2 + \frac{235}{98}\eta^2 - \frac{272}{147}\epsilon\eta \right) \\
R_0 &= \left(1 + \frac{6}{7}\epsilon - \frac{3}{7}\eta + \frac{125}{147}\epsilon^2 + \frac{25}{98}\eta^2 - \frac{62}{147}\epsilon\eta \right).
\end{aligned}\right\} \quad (1 \cdot 68)$$

---

* Bull. Geod., **38**, 1933, 178-87.

If W_0 denotes the value of the potential on the ellipsoid, we have

$$W_0 = \frac{1}{2} fM A_{00} + \frac{1}{2}\omega^2\Big[B_1 (A_{10} - c^2 A_{13}) + B_2 (A_{20} - c^2 A_{23}) \Big],$$

where A_{00} is the value of $V_0 (u)$ for $u = 0$.

Substituting the values of the various constants, we obtain

$$W_0 = \frac{fM}{a}\Big[1 + \frac{1}{3}\epsilon + \frac{1}{3} m + \frac{1}{3}\eta + \frac{2}{15}\epsilon^2 - \frac{1}{2} m^2 + \frac{1}{3} m\epsilon \Big].$$

By equation (1·62)

$$g_a = \frac{fM}{bc} - \frac{\omega^2}{a}\left(\frac{2 B_1}{abc} - E\right)$$

$$= \frac{fM}{a^2}(1 + \epsilon + \eta + \epsilon^2 + \eta^2 + \epsilon\eta) - \omega^2 a \left(\frac{3}{2} + \frac{3}{7}\epsilon + \frac{8}{7}\eta + \frac{125}{294}\epsilon^2 + \frac{80}{147}\eta^2 + \frac{46}{147}\epsilon\eta\right)$$

$$= \frac{fM}{a^2}\left(1 + \epsilon + \eta - \frac{3}{2} m + \epsilon^2 + \eta^2 + \frac{9}{4} m^2 + \epsilon\eta - \frac{27}{14} m\epsilon - \frac{37}{14} m\eta \right), \quad \ldots \quad (1·69)$$

where $m = \dfrac{\omega^2 a}{g_a} = \dfrac{\omega^2 a^3}{fM}\left(1 - \epsilon + \dfrac{3}{2}\dfrac{\omega^2 a^3}{fM}\right).$

The other two constants involved in equation (1·63) are given by

$$\left.\begin{aligned} p &= \frac{5}{2} m + \epsilon^2 - \frac{26}{7} m\epsilon - 2\epsilon + \frac{5}{7} m\eta \\ q &= -2\eta + \eta^2 + \frac{19}{7} m\eta. \end{aligned}\right\} \quad \ldots \quad (1·70)$$

In equation (1·65), the coefficients of the various terms inside the bracket, correct to order ϵ^2, are

$$\left.\begin{aligned} \frac{1}{2}(e_2^2 + 2p) &= -\epsilon + \frac{5}{2} m + \frac{1}{2}\epsilon^2 - \frac{26}{7} m\epsilon \\ \frac{1}{2}(e_1^2 + 2q) &= -\eta \\ \frac{1}{8} e_2^2 (3e_2^2 + 4p) &= \frac{5}{2} m\epsilon - \frac{1}{2}\epsilon^2. \end{aligned}\right\} \quad \ldots \quad (1·71)$$

Equation (1·65) may also be written as

$$g = g_a \left(1 - \frac{1}{2}\eta \right)\Big[1 + A' \sin^2\phi - B' \sin^2 2\phi + C' \cos^2\phi \cos 2L \Big]$$

$$= G' (1 + A' \sin^2\phi - B' \sin^2 2\phi + C' \cos^2\phi \cos 2L), \qquad (1·72)$$

where

$$\left.\begin{aligned}
A' &= -\epsilon + \frac{5}{2}m + \frac{1}{2}\eta - \frac{17}{14}m\,\epsilon \\
B' &= \frac{5}{8}m\,\epsilon - \frac{1}{8}\epsilon^2 \\
C' &= \frac{1}{2}\eta \\
G' &= g_a\left(1 - \frac{1}{2}\eta\right) \\
&= \frac{fM}{a^2}\left(1 + \epsilon + \frac{1}{2}\eta - \frac{3}{2}m + \epsilon^2 + \frac{9}{4}m^2 \right. \\
&\qquad\qquad \left. - \frac{27}{14}m\,\epsilon\right).
\end{aligned}\right\} \quad \ldots \quad (1\cdot73)$$

This is the form in which the formula for normal gravity is usually expressed, as we shall see in the next chapter.

It is interesting to compare formula (1 · 72) with the corresponding formula (1 · 15) for a static homogeneous triaxial ellipsoid. The two formulæ are similar. By a simple manipulation, equation (1 · 72) may be written as

$$g = \frac{fM}{a^2}\left(1 + D'' + A''\sin^2\phi - B''\sin^2 2\phi + C''\cos^2\phi\cos 2L\right), \quad (1\cdot74)$$

$$\left.\begin{aligned}
\text{where } D'' &= \epsilon + \frac{1}{2}\eta - \frac{3}{2}m + \epsilon^2 + \frac{9}{4}m^2 - \frac{27}{14}m\epsilon \\
A'' &= -\epsilon + \frac{5}{2}m + \frac{1}{2}\eta - \epsilon^2 - \frac{15}{4}m^2 + \frac{39}{14}m\epsilon \\
B'' &= \frac{5}{8}m\,\epsilon - \frac{1}{8}\epsilon^2 \\
C'' &= \frac{1}{2}\eta.
\end{aligned}\right\} \quad \ldots \quad (1\cdot75)$$

The values of the constants are now directly comparable with those of equation (1 · 15). We see that the coefficients are quite different in the two cases, the reason being that formula (1 · 72) pertains to a level surface, while (1 · 15) does not.

To see how far the static homogeneous triaxial ellipsoid deviates from a level surface, we will work out the angle χ which the gravity vector at a point (ϕ, L) on it makes with the normal. Obviously,

$$\cos\chi = \frac{\cos\phi\cos L\, g_x + \cos\phi\sin L\, g_y + \sin\phi\, g_z}{g}.$$

Substituting the values for g_x, g_y, g_z and g from equations (1 · 12) and (1 · 13) and simplifying, we have

$$\cos\chi = 1 - \frac{2}{25}\epsilon^2\sin^2 2\phi.$$

This equation shows that χ attains its maximum value at $\phi = 45°$. For $\epsilon = \frac{1}{297}$, this value amounts to about 5 min. of arc.

Another point to which attention might well be directed is to compare the expressions (1·7) and (1·52) for the external potentials of an ellipsoid. Omitting the term arising from the centrifugal force in equation (1·52) and bearing in mind that $V_0(u) = A_{00}$ for $u=0$, we see that this expression becomes identical with (1·7) if

$$B_1 = \frac{2}{3}\frac{\pi f \rho a^5}{\omega^2}(2\epsilon - 3\epsilon^2 - 2\epsilon\eta)$$

$$\text{and } B_2 = \frac{2}{3}\frac{\pi f \rho a^5}{\omega^2}(2\epsilon - 2\eta - 3\epsilon^2 + 3\eta^2).$$

These values of B_1, B_2 are different from the ones obtained in (1·67). A little consideration shows, however, that the orders of magnitude are the same. The discrepancy is due to the fact that B_1, B_2 of equation (1·67) appertain to a rotating ellipsoid, on which the potential is constant, while our present values correspond to an ordinary static ellipsoid.

We will now consider the important case when the level surface is an oblate spheroid. The appropriate expression for the external potential function in this case is

$$W = A_0 V_0 + A_1 V_1 + \frac{1}{2}\omega^2(x^2+y^2), \qquad \dots \quad (1\cdot76)$$

where

$$V_0 = \int_u^\infty \frac{ds}{\sqrt{\psi(s)}}, \quad V_1 = \int_u^\infty \left(1 - \frac{x^2+y^2}{a^2+s} - \frac{z^2}{c^2+s}\right)\frac{ds}{\sqrt{\psi(s)}}, \qquad (1\cdot77)$$

and $\psi(s) = (a^2+s)^2(c^2+s)$.

The condition $W = W_0$ for $u=0$ gives

$$\left.\begin{aligned} W_0 &= A_0 A_{00} + A_1(A_{00} - c^2 A_{30}) \\ A_0 &= \frac{1}{2} fM + \frac{1}{3}\omega^2 \frac{c^3(1+e'^2)}{(3+e'^2)\dfrac{(e'-\tan^{-1}e')}{e'^3} - 1} \\ A_1 &= -\frac{1}{2}\omega^2 \frac{c^3(1+e'^2)}{(3+e'^2)\dfrac{(e'-\tan^{-1}e')}{e'^3} - 1}, \end{aligned}\right\} \dots \quad (1\cdot78)$$

where $e'^2 = \dfrac{a^2-c^2}{c^2}$.

Proceeding in the same way as for a triaxial ellipsoid, we have

$$g = \frac{2A_0}{a^2}(1+e'^2\cos^2\phi)^{\frac{1}{2}} + 2cA_1(1+e'^2\cos^2\phi)^{-\frac{1}{2}} \times \frac{2}{e'^3c^3}(e' - \tan^{-1}e'). \qquad (1\cdot79)$$

If G_e and G_p denote the values of gravity at the equator and pole respectively, we have

$$\left.\begin{aligned} G_e &= \frac{2A_0}{a^2}(1+e'^2)^{\frac{1}{2}} + 2cA_1\left[\frac{2}{e'^3c^3}(e'-\tan^{-1}e')\right](1+e'^2)^{-\frac{1}{2}} \\ G_p &= \frac{2A_0}{a^2} + 2cA_1\left[\frac{2}{e'^3c^3}(e'-\tan^{-1}e')\right]. \end{aligned}\right\} \quad (1\cdot80)$$

Hence *

$$g = \frac{a\,G_e \cos^2\phi + c\,G_p \sin^2\phi}{(a^2\cos^2\phi + c^2\sin^2\phi)^{\frac{1}{2}}}$$

$$= G_e \frac{1 + p\sin^2\phi}{\sqrt{1 - e^2\sin^2\phi}}$$

$$= G_e \Big[1 + \frac{1}{2}(e^2 + 2p)\sin^2\phi + \frac{e^2}{2.4}(3e^2 + 4p)\sin^4\phi + \cdots$$

$$+ \frac{1.3\ldots\ldots(2n-3)}{2.4\ldots\ldots\;2n}\left\{(2n-1)\,e^2 + 2np\right\} e^{2n}\sin^{2n}\phi + \ldots \Big], \qquad (1\cdot81)$$

where $p = \dfrac{c\,G_p - a\,G_e}{a\,G_e}$ and $e^2 = \dfrac{a^2 - c^2}{a^2}$. ... ... $(1\cdot82)$

The values of the various constants have been worked out in terms of the ellipticity ϵ with the following results :

$$\left.\begin{aligned}
A_0 &= \frac{1}{2} fM - \frac{5}{8}\omega^2 a^3 \left(\frac{1}{\epsilon} - \frac{11}{14} + \frac{23}{196}\epsilon\right) \\
&= \frac{1}{2} fM \left[1 - \frac{5}{4}\frac{m}{\epsilon}\left(1 - \frac{3}{2}m + \frac{9}{4}m^2 - \frac{3}{4}m\epsilon + \frac{3}{14}\epsilon + \frac{65}{196}\epsilon^2\right)\right] \\
A_1 &= \frac{15}{16}\omega^2 a^3 \left[\frac{1}{\epsilon} - \frac{11}{14} + \frac{23}{196}\epsilon\right] \\
&= \frac{15}{16} fM \frac{m}{\epsilon}\left(1 - \frac{3}{2}m + \frac{9}{4}m^2 - \frac{3}{4}m\epsilon + \frac{3}{14}\epsilon + \frac{65}{196}\epsilon^2\right) \\
m &= \frac{\omega^2 a}{G_e} = \frac{\omega^2 a^3}{fM}\left(1 - \epsilon + \frac{3}{2}\frac{\omega^2 a^3}{fM}\right).
\end{aligned}\right\} \qquad (1\cdot83)$$

$$\left.\begin{aligned}
W_0 &= \frac{fM}{a}\left(1 + \frac{1}{3}\epsilon + \frac{1}{3}m + \frac{2}{15}\epsilon^2 - \frac{1}{2}m^2 + \frac{1}{3}m\epsilon\right) \\
G_e &= \frac{fM}{ac} - \frac{3}{2}\omega^2 a\left(1 + \frac{2}{7}\epsilon\right) \\
&= \frac{fM}{a^2}\left(1 + \epsilon - \frac{3}{2}m + \frac{9}{4}m^2 + \epsilon^2 - \frac{27}{14}m\epsilon\right) \\
G_p &= \frac{fM}{a^2} + \omega^2 a\left(1 - \frac{1}{7}\epsilon\right) \\
&= \frac{fM}{a^2}\left(1 + m + \frac{6}{7}m\epsilon - \frac{3}{2}m^2\right) \\
p &= \frac{5}{2}m - 2\epsilon + \epsilon^2 - \frac{26}{7}m\epsilon\,.
\end{aligned}\right\} \quad \ldots \qquad (1\cdot84)$$

Equation $(1\cdot81)$ may also be written in the form†

$$g = G_e\,(1 + A'\sin^2\phi - B'\sin^2 2\phi - B_2\sin^2\phi\sin^2 2\phi - B_3\sin^4\phi\sin^2 2\phi - \ldots\ldots\ldots), \qquad (1\cdot85)$$

where

$$\left.\begin{aligned}
A' &= \frac{G_p - G_e}{G_e} = \frac{5}{2}m - \epsilon - \frac{5}{2}m\left(\frac{17}{35}\epsilon + \frac{1}{245}\epsilon^2 + \frac{13}{18865}\epsilon^3 + \ldots\right) \\
B' &= \frac{1}{8}\epsilon\,(\epsilon + 2A') \\
B_2 &= \frac{1}{8}\epsilon^2\left(2\epsilon + 3A'\right) - \frac{1}{32}\epsilon^3\left(3\epsilon + 4A'\right), \text{ etc.}
\end{aligned}\right\} \qquad (1\cdot86)$$

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* **Bull. geod. 38, 1933, 178–87.**
† **Ibid 25, 1930, 40–49.**

This is an extension of equation (1·35) based on Helmert's theory. Helmert included only terms up to the second order in ϵ, and considered the first three terms of the series expansion. Equation (1·85) can be written down to any number of terms that we like, the law of coefficients being known. The same applies to formula (1·65) for the case of an ellipsoid with unequal axes, which can easily be extended to include higher order terms. For computing theoretical gravity, however, the normal gravity formulæ usually take into account only terms up to order ϵ^2 (see chap. II).

Sphere:—The case of a spherical level surface is capable of easy solution. The potential

$$W = \frac{fM}{r} + \frac{1}{2}\,\omega^2 r^2 \cos^2\theta + U_0$$

must be a constant on the sphere. This suggests that U_0 should be of the form $\frac{AP_2}{r^3}$. The boundary condition that W is a constant for $r = a$ determines the constant A, and gives

$$W = \frac{fM}{r} + \frac{\omega^2 a^5}{3r^3} P_2 + \frac{1}{2}\,\omega^2 r^2 \cos^2\theta. \qquad \ldots \quad (1\cdot 87)$$

The surface being level, the resultant gravity is along the normal, and is given by

$$g = -\frac{\delta W}{\delta r} = \frac{fM}{r^2} + \frac{\omega^2 a^5}{r^4} P_2 - \omega^2 r \cos^2\theta$$

$$= \frac{fM}{a^2} + \omega^2 a\, P_2 - \omega^2 a \cos^2\theta. \qquad \ldots \quad \ldots \quad (1\cdot 88)$$

7. Clairaut's equation.—Neglecting the longitude term in equation (1·25), the value of gravity on the level spheroid becomes

$$g = \frac{fM}{r^2}\left\{1 + \frac{3K}{2r^2} - \frac{\omega^2 r^3}{fM} + \left(\frac{\omega^2 r^3}{fM} - \frac{9K}{2r^2}\right)\sin^2\theta\right\}.$$

This expression for gravity is deduced from the equation (1·25) for the potential by assuming $A = B$. If a, c denote the lengths of the equatorial and polar semi-axes, and G_e, G_p the corresponding values of gravity at their extremities, we have

$$\left.\begin{aligned} G_e &= \frac{fM}{a^2}\left(1 + \frac{3K}{2a^2} - \frac{\omega^2 a^3}{fM}\right) \\ G_p &= \frac{fM}{c^2}\left(1 - \frac{3K}{c^2}\right). \end{aligned}\right\}$$

Putting $\epsilon = \frac{a-c}{a}$, and $K = \frac{1}{M}(C - A)$ we have

$$\frac{G_e - G_p}{G_e} = \epsilon + \epsilon^2 + \epsilon m - \frac{5}{2} m$$

$$\doteqdot \epsilon - \frac{5}{2}\frac{\omega^2 a}{G_e}. \qquad \ldots \quad (1\cdot 89).$$

This relation between the values of gravity at the equator and the pole is correct to first order terms in ϵ, and is known as Clairaut's equation.

For a true spheroid, the above may be deduced by eliminating A_0 between the two equations (1·80). We get

$$G_e - \frac{a}{c} G_p = 2cA_1 \left[\frac{2}{e'^3 c^3} (e' - \tan^{-1} e') \right] (1 + e'^2)^{-\frac{1}{2}}$$
$$- 2cA_1 \left[\frac{2}{e'^3 c^3} (e' - \tan^{-1} e') \right] (1 + e'^2)^{\frac{1}{2}},$$

which when simplified leads to equation (1·89).

It might be noted that the ellipticity ϵ and the difference $(G_p - G_e)$ vary in opposite directions. The value of $(G_p - G_e)$ is maximum for a spherical level surface. By (1·88) we see that for such a surface it is given by

$$\frac{G_p - G_e}{G_e} = \frac{5}{2} \frac{\omega^2 a}{G_e}.$$

Another relation which follows from equations (1·84) is

$$\frac{2 G_e}{a} + \frac{G_p}{c} = 4\pi f \rho_m - 2\omega^2,$$

where ρ_m denotes the mean density of the matter inside the spheroid.

8. Values of gravity at the extremities of the principal axes of a triaxial ellipsoid.—For a triaxial ellipsoid, the relations between the values of gravity at the extremities of the principal axes can be derived from equations (1·62). Denoting these by (g_a, g_b, g_c), we have

$$\left.\begin{aligned}
\frac{g_b}{b} - \frac{g_c}{c} &= -\frac{2\omega^2}{abc} \cdot \frac{B_2}{b^2} + \omega^2 \left(\frac{1}{b^2} - \frac{1}{c^2} \right) E \\
&= -\frac{5}{2} \omega^2 \left(1 + \frac{18}{35} \epsilon - \frac{4}{5} \eta \right), \\
\frac{g_c}{c} - \frac{g_a}{a} &= \frac{2\omega^2}{abc} \cdot \frac{B_1}{a^2} + \omega^2 \left(\frac{1}{c^2} - \frac{1}{a^2} \right) E \\
&= \frac{5}{2} \omega^2 \left(1 + \frac{18}{35} \epsilon + \frac{2}{7} \eta \right), \\
\frac{g_a}{a} - \frac{g_b}{b} &= \frac{2\omega^2}{abc} \left(\frac{B_2}{b^2} - \frac{B_1}{a^2} \right) + \omega^2 \left(\frac{1}{a^2} - \frac{1}{b^2} \right) E \\
&= -\frac{19}{7} \omega^2 \eta.
\end{aligned}\right\} \quad \dots \quad (1·90)$$

The relative orders of magnitude of (g_a, g_b, g_c) can also be obtained from equations (1·62) by substitution of the values for B_1 and B_2. After some simplification, we have

$$g_a = \frac{fM}{a^2} \left(1 + \epsilon + \eta - \frac{3}{2} m + \epsilon^2 + \eta^2 + \frac{9}{4} m^2 + \epsilon\eta - \frac{27}{14} m\epsilon - \frac{37}{14} m\eta \right),$$

$$g_b = \frac{fM}{a^2} \left(1 + \epsilon - \frac{3}{2} m + \epsilon^2 - \frac{27}{14} m\epsilon + \frac{9}{4} m^2 + \frac{11}{7} m\eta \right),$$

$$g_c = \frac{fM}{a^2} \left(1 + \eta + m + \eta^2 - \frac{3}{2} m^2 + \frac{6}{7} m\epsilon + \frac{4}{7} m\eta \right).$$

Hence

$$g_a - g_b = \frac{fM}{a^2}\eta\left[1+\epsilon-\frac{59}{14}m+\eta\right], \quad \ldots \quad (1\cdot 91)$$

$$g_c - g_a = \frac{fM}{a^2}\left[\frac{5}{2}m-\epsilon-\epsilon^2-\frac{15}{4}m^2-\epsilon\eta+\frac{39}{14}m\epsilon+\frac{45}{14}m\eta\right].(1\cdot 92)$$

For the values of ϵ, m appropriate to the earth, the above two equations show that $g_c > g_a > g_b$, an important result, which is by no means obvious at first sight. g_a will be less than g_b, if

$$\frac{59}{14}m-\epsilon > 1 \text{ or if } \omega^2 > 4\times 10^{-7}\ \text{sec}^{-2}.$$

This will occur for a rotating body whose period of rotation is less than 2·8 hours.

9. Summary.—We have discussed above, different methods of determining the gravity formulæ. Helmert's method consists in selecting some special terms from the general expression (1·24) for the potential. This modified potential defines his level spheroid and leads to the value of gravity on it. Darwin's method is practically identical.

Pizetti's method gives rigorous expressions for gravity on an equipotential sphere, spheroid and triaxial ellipsoid. These formulæ can be expanded in series, and we can take as many terms as are necessary for the accuracy aimed at. To obtain terms beyond the third by Helmert's method requires great labour. In formula (1·85) for example, five terms are given, while the corresponding Helmert's formula gives three terms. Again, in Helmert's method, in the expression for the coefficients of the gravity formula, terms beyond order ϵ^2 are neglected as they involve laborious calculations, while Pizetti's method includes higher order terms.

Given the dimensions of a spheroid, the constants of the gravity formula appertaining to it can be derived to any accuracy that we like by Pizetti's method, but not by Helmert's and Darwin's methods. For example, for the International spheroid, values of normal gravity can be computed to four places of decimals by the formula $\gamma_0 = 978\cdot 049\ (1 + 52884\times 10^{-7}\sin^2\phi - 59\times 10^{-7}\sin^2 2\phi)$. When however 6-figure accuracy is wanted, one more term as given by Pizetti's formula (1·85) is required. It might be remarked, however, that with the present degree of accuracy of gravity measurements, the values of normal gravity to six decimal places are of academic interest only. For all practical purposes, Helmert's formulæ are good enough.

Stokes' formula (1·19) for gravity is based on first order considerations, and is not accurate enough to be used either for normal gravity, or for the determination of ellipticity. But we shall see in chapter v that his formula is very valuable for determining the undulations of the geoid with respect to a suitably chosen reference surface.

CHAPTER II

GRAVITY FORMULÆ AS OBTAINED IN PRACTICE, AND THEIR COMPARISON WITH THE THEORETICAL FORMULÆ

1. Method of deriving gravity formulæ.—We see from the theoretical considerations of the preceding chapter how we can determine the form of a level surface from the variations of gravity on it. The surface of the earth is not an equipotential, and to make the above theory applicable, the observed values of gravity are reduced to the geoid by a suitable reduction. If the geoid were a true spheroid, observations of gravity at three known points would enable us to define it. If it were a triaxial ellipsoid, knowledge of gravity at four points would be required. In practice, however, to obtain reliable values of the constants in gravity formula, it is customary to make use of all the available gravity data and apply the method of least squares. The geoidal values of gravity have been fitted to formulæ of the type

$$g = G_e\ (1 + A' \sin^2\phi - B' \sin^2 2\phi)$$

and $g = G'\ [1 + A' \sin^2\phi - B' \sin^2 2\phi + C' \cos^2\phi \cos 2 (L - L_0)]$ by various investigators, and the results are tabulated in the next para. As we have seen already, these formulæ take account of only terms up to the second order in the ellipticity ϵ.

Ackerl *, using Prey's reduction, has expressed the gravity field of the earth in terms of spherical harmonic functions up to the 16th order. He undertook these laborious calculations with a view to utilising them for finding the undulations of the geoid. We shall see in chap. v that this has not proved a fruitful field of research as Prey's anomalies are unsuitable for this purpose.

2. Gravity formulæ.—The following are the main gravity formulæ which have been obtained at various times. The relevant values of ellipticity of the level surface, and the constant χ defining its departure from a true spheroid (see page 15) are given below each formula. The relation between χ and the coefficient of $\sin^2\phi$ in the gravity formulæ will be explained in the next para. γ_0 denotes normal gravity.

(1) Helmert 1901.

$$\gamma_0 = 978 \cdot 030\ [1 + 5302 \times 10^{-6} \sin^2\phi - 7 \times 10^{-6} \sin^2 2\phi],$$

$$\epsilon = \frac{1}{298 \cdot 3}, \chi = -205 \times 10^{-8}.$$

* Akad. Wien, sitz.-ber. d. mathem. naturw. kl. (II *a*), **140**, 1931 and **141**, 1932.

(2) Helmert 1915.

$$\gamma_0 = 978 \cdot 052 \ [\, 1 + 5285 \times 10^{-6} \sin^2\phi - 7 \times 10^{-6} \sin^2 2\phi + 18 \times 10^{-6} \cos^2\phi \cos 2(L + 17^\circ) \,],$$

$$\epsilon = 0 \cdot 003370 + 18 \times 10^{-6} \cos 2(L + 17^\circ),$$

$$\text{average } \epsilon = \frac{1}{296 \cdot 7}, \ \chi = -205 \times 10^{-8}.$$

(3) Berroth 1916.

$$\gamma_0 = 978 \cdot 046 \ [\, 1 + 5296 \times 10^{-6} \sin^2\phi - 7 \times 10^{-6} \sin^2 2\phi + 11 \cdot 6 \times 10^{-6} \cos^2\phi \cos 2(L + 10^\circ) \,],$$

$$\epsilon = 0 \cdot 003358 + 11 \cdot 6 \times 10^{-6} \cos 2(L + 10^\circ),$$

$$\text{average } \epsilon = \frac{1}{297 \cdot 8}, \ \chi = -205 \times 10^{-8}.$$

(4) Bowie 1917.

$$\gamma_0 = 978 \cdot 039 \ [\, 1 + 5294 \times 10^{-6} \sin^2\phi - 7 \times 10^{-6} \sin^2 2\phi \,],$$

$$\epsilon = \frac{1}{297 \cdot 4}, \ \chi = -205 \times 10^{-8}.$$

(5) Survey of India spheroid II.

$$\gamma_0 = 978 \cdot 021 \ [\, 1 + 5236 \times 10^{-6} \sin^2\phi - 6 \times 10^{-6} \sin^2 2\phi \,],$$

$$\epsilon = \frac{1}{292 \cdot 4}, \ \chi = 0.$$

(6) The best formula for India, as available from data till 1929 is

$$\gamma_0 = 978 \cdot 021 \ [\, 1 + 5359 \times 10^{-6} \sin^2\phi - 6 \times 10^{-6} \sin^2 2\phi \,],$$

$$\epsilon = \frac{1}{301}, \ \chi = 0.$$

(7) Heiskanen 1924 (without longitude term).

$$\gamma_0 = 978 \cdot 048 \ [\, 1 + 5293 \times 10^{-6} \sin^2\phi - 7 \times 10^{-6} \sin^2 2\phi \,],$$

$$\epsilon = \frac{1}{297 \cdot 4}, \ \chi = -205 \times 10^{-8}.$$

(8) Heiskanen 1924 (with longitude term).

$$\gamma_0 = 978 \cdot 052 \ [\, 1 + 5285 \times 10^{-6} \sin^2\phi - 7 \times 10^{-6} \sin^2 2\phi + 27 \times 10^{-6} \cos^2\phi \cos 2 (L - 18^\circ) \,],$$

$$\epsilon = 0 \cdot 003370 + 27 \times 10^{-6} \cos 2(L - 18^\circ),$$

$$\text{average } \epsilon = \frac{1}{296 \cdot 7}, \ \chi = -205 \times 10^{-8}.$$

(9) Heiskanen 1928.

$$\gamma_0 = 978 \cdot 049 \ [\, 1 + 5293 \times 10^{-6} \sin^2\phi - 7 \times 10^{-6} \sin^2 2\phi + 19 \times 10^{-6} \cos^2\phi \cos 2 (L - 0^\circ) \,],$$

$$\epsilon = 0 \cdot 003364 + 19 \times 10^{-6} \cos 2 (L - 0^\circ),$$

$$\text{average } \epsilon = \frac{1}{297 \cdot 3}, \ \chi = -205 \times 10^{-8}.$$

(10) Heiskanen 1938.

$$\gamma_0 = 978 \cdot 052 \ [\, 1 + 5297 \times 10^{-6} \sin^2\phi - 59 \times 10^{-7} \sin^2 2\phi + 28 \times 10^{-6} \cos^2\phi \cos 2 (L + 25^\circ) \,],$$

$$\epsilon = 0 \cdot 003358 + 28 \times 10^{-6} \cos 2 (L + 25^\circ),$$

$$\text{average } \epsilon = \frac{1}{297 \cdot 8}, \ \chi = 0.$$

(11) De Sitter 1927.

$$\gamma_0 = 978\cdot052\ [1+52884\times10^{-7}\sin^2\phi - 75\times10^{-7}\sin^2 2\phi],$$

$$\epsilon = \frac{1}{296\cdot96\pm0\cdot10}, \chi = -205\times10^{-8}.$$

(12) International spheroid.

$$\gamma_0 = 978\cdot049\ [1+52884\times10^{-7}\sin^2\phi - 59\times10^{-7}\sin^2 2\phi],$$

$$\epsilon = \frac{1}{297}, \quad \chi = 0.$$

(13) Jeffreys 1936.

$$\gamma_0 = 978\cdot051\ [1+5282\times10^{-6}\sin^2\phi - 7\times10^{-6}\sin^2 2\phi],$$

$$\epsilon = \frac{1}{296\cdot38\pm0\cdot51}, \chi = -205\times10^{-8}.$$

3. Interpretation of the constants.—The constant B′ in the formula $g = G_e\ (1 + A' \sin^2\phi - B' \sin^2 2\phi)$ is a small quantity, its magnitude being about 1/800 times that of A', and it has been found that it cannot be deduced by least square solution with any accuracy. Its value has to be assigned in some other way. By Helmert's theory, $B' = -\frac{1}{8}\epsilon^2 + \frac{5}{8} m\epsilon - \frac{3}{4}\chi$ for a level spheroid. The value of χ has been inferred by Darwin from two quite different assumptions about the internal constitution of the earth. He first assumed Roche's law of density, and obtained $\chi = -205\times10^{-8}$. Then he used Wiechert's law that the earth consists of a solid core of density 8·206, on which is superposed a mantle of density 3·2, and deduced $\chi = -175\times10^{-8}$. Taking $\chi = -205\times10^{-8}$, $m = \frac{1}{288\cdot41}$ and $\epsilon = \frac{1}{298}$, we get $B' = 7\times10^{-6}$.

The quantity $7\times10^{-6}\sin^2 2\phi$ which occurs in most of the gravity formulæ is thus based on theoretical considerations. The magnitude of this term is $7\times10^3\times10^{-6}\sin^2 2\phi$ gals. The maximum value that it can attain is 0·007 gals, which is quite appreciable. In India the magnitude of this term ranges from 0·0005 to 0·0063 gals. Darwin's work shows that the figure 7×10^{-6} for B' is quite insensitive to the hypothesis about the internal constitution of the earth. It appertains to a level spheroid, depressed below an exact spheroid by about 10 feet in latitude 45°.

For a true spheroid, we see from (1·36) that $B' = \frac{1}{8}\epsilon\ (5m-\epsilon)$. Knowing m and ϵ we can get the value of B′. As an example we have $m = \frac{1}{288\cdot36}$, $\epsilon = \frac{1}{297}$ for the International spheroid. These values give $B' = 5869\times10^{-9}$.

From the foregoing discussion, we infer that if the value of B′ is assigned in a normal gravity formula, it implies a radial departure of the level spheroid from a true spheroid having the same ellipticity by

$$\frac{4}{3} a \sin^2\theta\cos^2\theta\left(-\frac{1}{8}\epsilon^2 + \frac{5}{8}m\epsilon - B'\right).$$

The remaining constants in the gravity formula are derived from least square solution. Helmert's* 1915 formula was deduced by him from 3000 stations reduced by free-air. Formulæ (7) and (8) were computed by Heiskanen† with the aid of 656 stations, the one with the longitude term and the other without it. All stations in the same degree sheet were treated as a single station. In other words, he used 656 degree squares on the globe. In 1928, using 841 squares and including the longitude term‡ he obtained formula (9), while without the L-term he obtained

$$\gamma_0 = 978\cdot044\ [\ 1 + 5301 \times 10^{-6} \sin^2\phi - 7 \times 10^{-6} \sin^2 2\phi\ ]. \quad \ldots(2\cdot1)$$

Formula (10) was based on much more data, there being 1591 squares. Heiskanen used isostatically reduced values of gravity and omitted the stations on islands and ocean deeps where the isostatic anomalies are large. He also did not consider the squares in the Red Sea since the anomalies there are all highly positive.

G_e denotes the equatorial value of gravity. Its values found by different authors utilizing different observation material range from 978·030 to 978·052. The differences are partly due to the values of gravity being reduced to the geoid in different ways and partly due to the different location and extent of the gravity data. The spread of the gravity observations used is probably responsible for a greater part of the discrepancy.

When the longitude term is included in the gravity formula, G_e has to be replaced by G' which denotes the mean value of the equatorial gravity. According to Heiskanen's latest formula,

$$G' = 978\cdot052.$$

The use of different gravity formulæ by different countries is obviously undesirable. The adoption of the International spheroid by the International Union of Geodesy and Geophysics at the Madrid meeting in 1924 as a standard basis for astronomico-geodetic work also led to the demand for a universal gravity formula. The first step towards this objective was to fix a value for G_e. Silva§ suggested that $G_e = 978\cdot049$ was the best value, as it would secure the best agreement with the observed values of gravity.

The dimensions of the International spheroid are

$$\left.\begin{aligned} a &= 6{,}378{,}388 \text{ metres} = 20{,}926{,}488\cdot03 \text{ feet}, \\ \epsilon &= \frac{1}{297}. \end{aligned}\right\}$$

Substituting $G_e = 978\cdot049$, $\omega = \dfrac{2\pi}{86164\cdot1} = 7{,}292{,}115 \times 10^{-11}$ and the above values of a, ϵ in (1·36), we have

$$\left.\begin{aligned} A' &= 5{,}288{,}384 \times 10^{-9} \\ B' &= 5{,}869 \times 10^{-9}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2\cdot2)$$

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* Sitzungsberichte der K. Preu. Akad. der Wiss. **41**, 1915, 676–85.
† Veroff. de Finnischen Geod. Inst. **4**, 1924, chap. III.
‡ Heiskanen. Gerl. Beit. z. Geoph. **19**, 1928, 356–77.
§ Accad.-Nazionale dei Lincei, 1930.

In the International formula (12) therefore, G_e is deduced from the various least square solutions, and A', B' from theoretical expressions for gravity on a spheroid.

An important fact about G_e, which is worth mentioning, is that it fixes the mass, and hence the mean density of the equipotential surface. Thus for a spheroid (a, ϵ) we have from equation (1·84)

$$\frac{G_e}{a} = \frac{4}{3}\pi f \rho_m - \frac{3}{2}\omega^2\left(1 + \frac{2}{7}\epsilon\right), \quad \ldots \quad (2\cdot 3)$$

where ρ_m denotes the mean density of the matter inside the spheroid. Taking $G_e = 978\cdot049$, $\omega^2 = 0\cdot5256 \times 10^{-8}$ and $f = 6\cdot675 \times 10^{-8}$, the above equation gives the value of ρ_m for the International spheroid $\left(a = 6\cdot378388 \times 10^8 \text{ cm.}, \epsilon = \frac{1}{297}\right)$ to be

$$\rho_m = 5\cdot5124 \text{ gm./cm}^3.$$

The corresponding value in the case of the Everest spheroid $\left(a = 6\cdot377276 \times 10^8 \text{ cm.}, \epsilon = \frac{1}{300\cdot8}\right)$ is

$$\rho_m = 5\cdot5133 \text{ gm./cm}^3.$$

The constant A' is very important, as it gives a clue to the ellipticity ϵ of the level surface. From formula (1·36) we see that it depends on both ϵ and m, where $m = \frac{\omega^2 a}{G_e}$. In other words, the value of this constant depends on the value chosen for the equatorial gravity. The variation is small however. A change of 1×10^{-3} in G_e corresponds to a change of $8\cdot8 \times 10^{-9}$ in A'.

The following table gives the variation of A' with ϵ for the value of $m = \frac{1}{288\cdot361}$, which corresponds to $G_e = 978\cdot049$ and $a = 6{,}378{,}388$ metres.

$\frac{1}{\epsilon}$	A'	$\frac{1}{\epsilon}$	A'
290	$10^{-6} \times 5207$	296	$10^{-6} \times 5277$
291	$10^{-6} \times 5219$	297	$10^{-6} \times 5288$
292	$10^{-6} \times 5231$	298	$10^{-6} \times 5300$
293	$10^{-6} \times 5242$	299	$10^{-6} \times 5311$
294	$10^{-6} \times 5254$	300	$10^{-6} \times 5322$
295	$10^{-6} \times 5266$		

Adopting the values of G_e and B' as given above, this table enables the formula for normal gravity to be written for any given spheroid. As an example, we know that for Clarke's 1880 spheroid, $\epsilon = \frac{1}{293 \cdot 5}$. The value of A' corresponding to this is 5248×10^{-6}. The expression for gravity on Clarke's spheroid is therefore

$$\gamma_0 = 978 \cdot 049 \ (1 + 5248 \times 10^{-6} \sin^2 \phi - 6 \times 10^{-6} \sin^2 2\phi).$$

We see that a change of 12×10^{-6} in A' corresponds to a change of 1 in $\frac{1}{\epsilon}$. This enables us to decide what order terms should be retained in the expression for A'. If we are content to obtain $\frac{1}{\epsilon}$ to one place of decimal, it suffices to retain terms up to order ϵ^2 ($= 11 \times 10^{-6}$). Terms of order ϵ^3 amount to 3×10^{-8}, and will have no effect on the first place of decimal in $\frac{1}{\epsilon}$. Stokes' formula (1·19), based on first order considerations, implies that $A' = \frac{5}{2}m - \epsilon$. The value of the reciprocal of the ellipticity deduced from this relation may be wrong by one unit.

Another method of determining the ellipticity which does not involve least squares is to make use of the equation (1·29), namely

$$\frac{3K}{2a^2} = \epsilon - \frac{1}{2} m - \epsilon^2 + \frac{1}{2} \epsilon m + \frac{3}{4} m^2 + \frac{1}{7} \delta.$$

The values of the constants K and δ are assigned from certain considerations, which will be dealt with in the next chapter, and the value of ϵ is deduced therefrom.

The range of ϵ evidenced by the gravity formulæ varies from $\frac{1}{292}$ to $\frac{1}{298}$. Like G_e this range is due to the use (or availability) of different areas of gravity survey. The value of ϵ deduced from gravity data depends of course on the gravity reduction employed.

4. Tables for normal gravity.—G. Cassinis* has published tables giving the values of gravity on the International spheroid, correct to three and four places of decimals. For this accuracy the formula $\gamma_0 = 978 \cdot 049 \ (1 + 52884 \times 10^{-7} \sin^2 \phi - 59 \times 10^{-7} \sin^2 2\phi)$ is sufficient. W. D. Lambert and F. W. Darling† have tabulated these values to 6 places of decimals. For this they had to include one more term, involving $\sin^6 \phi$, in the above formula. As mentioned at the end of the preceding chapter, this is a far higher accuracy than what is required in practice.

The change of normal gravity with ϕ is given by $\frac{\delta \gamma_0}{\delta \phi} \doteqdot G_e A' \sin 2\phi$. The change is most rapid at mid-latitudes (about 0·1 mgal per 1° latitude), and is zero at the equator and the pole.

5. The Longitude term and its significance.—Formulæ (2, 3, 8, 9 and 10) of para 2 are important, as they contain a

* Bull. Geod. 1931, 313. He also gives a table for converting gravity from International to Helmert's 1901 and Bowie's 1917 formulæ.

† Bull. Geod. 1931, 327.

longitude term which has provoked considerable discussion. From equation (1·22), we see that the coefficient of this term is $\frac{1}{2}\eta$, where η denotes the ellipticity of the equator. The difference $(a-b)$ between the semi-equatorial axes, the equatorial ellipticity η and the positions of the principal axes as given by the various formulæ are tabulated below:

		$a-b$ in metres	η	Longitude L_0 of major axis
Helmert	1915	230 ± 40	36×10^{-6}	$-17^\circ \pm 6^\circ$
Berroth	1916	150 ± 60	23×10^{-6}	-10
Heiskanen	1924	345 ± 40	54×10^{-6}	-18 ± 5
Heiskanen	1928	242 ± 40	38×10^{-6}	0 ± 5
Heiskanen	1938	352 ± 30	56×10^{-6}	-25 ± 2
Hirvonen	1933	139 ± 16	22×10^{-6}	-19 ± 3

L_0 denotes the longitude of one end of the major axis, reckoned positive east of Greenwich meridian.

It may be pointed out that the probable error of Hirvonen's 1933 result is least, not because it is the best determination, but because he used only a few points for its deduction.

Several attempts have been made to determine the ellipticity of the equator from arc measurements. As early as 1861 Clarke using three long arcs, the Russian, Franco-English and Indian, found by a least square solution the values for the difference of equatorial semi-axes and the position of major axis of the equator to be

$$a-b = 1620 \text{ metres}, \; L_0 = -14^\circ.$$

Using arc measurements in Europe and the United States, Heiskanen* found

$$a-b = (165 \pm 57) \text{ metres}, \; L_0 = +38^\circ \pm 10^\circ.$$

It has been contended by some geodesists†, that the actual geoid has a circular equator and that the longitude term in the gravity formulæ is introduced spuriously by the reduction employed, namely free-air or isostatic. Mader, by laborious computations, proved that if the average height of the continents be taken as 0·8 km., they would produce $a-b = 268$ metres if they were uncompensated, and $a-b = 278$ metres if they were compensated. His figure of 278 metres for the compensated geoid is however not correct‡, as he wrongly applied topographic reduction twice in his working.

Jung§ has worked out the effect of different mass types in producing the difference $(B-A)$ of the principal equatorial moments of inertia of the earth. A brief summary of his results is interesting. Taking the earth to be non-isostatic, the effect of

* Heiskanen, Veröffen. des Finnischen Geodät. Inst. No. 12. 1929.
† Mader, Gerl. Beit. Z. Geoph. **18**, 1927, 145–184.
Hopfner, „ „ „ **20**, 1928.
‡ Heiskanen, ibid.
§ Jung, Zeit. f. Geoph. Jahr. 4, Heft. 1.

superposing the continents and oceans on a homogeneous spherical earth is to produce $(B-A) = 5\cdot 2 \times 10^{40}$ c. g. s. The corresponding difference in the equatorial semi-axes is $(a-b) = 200$ metres, and $L_0 = 86°$. In other words, the superposition of continents and oceans as a load on a homogeneous spherical earth produces a longitude term $31 \times 10^{-6} \cos 2\,(L - 86°)$ in gravity. In this computation, the continents are assumed to be of uniform height 0·8 km. and density 3·2, and the oceans of uniform depth 0·4 km. and density 2·2.

For compensated continents and oceans, the corresponding results on the assumption of a depth of compensation of 80 km. are

$$B-A = 13 \times 10^{38} \text{ c. g. s., } a-b = 5 \text{ metres and } L_0 = 86°.$$

Hence the $(a-b)$ of the compensated geoid can differ only by about 5 or 10 metres from that of the actual geoid, and this is a very small fraction of the amount suggested by the gravity formulæ. Prey * also obtains the same result for the ellipticity of the equator produced by isostatic mass transfers.

The free-air and condensation reductions have also a negligible effect. Hence it is impossible to explain the values of $(B-A)$ derived from gravity formulæ by assuming that these are fallaciously introduced by the reductions employed.

Several types of density inequalities can be postulated to account for the longitude term. Schweydar has pointed out that a difference in density of 0·01 between layers 200 km. thick under the Atlantic and Indian Oceans would produce a systematic term of this type with amplitude 36×10^{-6}.

Similarly, Berroth † has shown that if the highlands of Central Asia (considered as bounded by $\phi = 25°$ to $50°$, and $L = 80°$ to $110°$) had a defect in density of 0·20 up to the depth of compensation, they would produce the above L-term. Of course, the actual observations in this region do not show such a large defect of density. The above only gives an idea of how much mass anomaly is needed.

Again, consider a sphere with a surface coating on it equivalent to a thickness Y_2 of material of normal crustal density 2·7. We shall see in chap. IV, para 7, that if this inequality is compensated according to Airy's hypothesis at a depth of compensation 35 miles, then $\Delta g = 0\cdot 0013$ gals for $Y_2 = 1$ mile. If this inequality were uncompensated, Δg would be 0·07 gals. For an L-term with an amplitude of 0·02 gals to be possible for an isostatic crust floating on a substratum, the solid surface of the earth must deviate from a spheroid of equal volume by about 15 miles, which we do not find to be the case in nature. If we assume this harmonic to be due to departures from isostasy, it can easily be shown that these are equivalent to a surface coating of thickness of about 2,000 feet of normal crustal density. Wide-spread

* Prey, Gerl. Beit. z Geoph. **36**, 1932, 242–68.
† Berroth, Gerl. Beit. z. Geoph. **14**, 1916, 245.

inequalities of such an extent lead to important physical implications. They produce considerable stresses which have to be borne by the rocks of earth's crust. The problems of the possible magnitude of stresses due to visible surface inequalities and the strength of the earth's crust are of fundamental importance, and have been considered by Darwin* and more recently in a thorough manner by Jeffreys.† There is no means of knowing the exact stress distribution inside the earth, as an infinite number of stress distributions can be found which will support the surface inequalities.

We shall see in chap. IV, para 6, that the gravity anomalies in India point to the existence of regions where there are departures from isostasy equivalent to a thickness of about 2,000 feet of surface rock. In the light of Jeffreys' work such loadings can easily be supported by the crust. Difficulties arise, however, in explaining the support of loads of considerable horizontal extent as implied in Heiskanen's longitude term. Jeffreys concludes that the mechanism of compensation (implying, as it does, hydrostatic conditions under the crust) demands a greater strength in the upper layers than we should need without compensation. Wide-spread inequalities of 2,000 feet of surface rock produce stresses which would require an impossible strength if they were to be supported by the upper layers alone. They require a strength in the lower layer (i.e. below 50 km.), which controverts the popular belief that below 50 km. there is hydrostatic equilibrium. This is also borne out by the phenomenon of deep-focus earthquakes. If materials at depths beyond 50 km. were entirely devoid of strength, it would not be possible for them to accummulate stresses, the release of which is essential for the production of an earthquake. An alternative mechanism for the support of these loadings of the crust has been brought forward by Meinesz.‡ It is based on the hypothesis that the disturbances of equilibrium are adjusted by convection currents in the substratum.

It has sometimes been argued against the longitude term that the level surface may be a spheroid, but being heterogeneous the principal equatorial moments of inertia may be different. In this case there would be a $(B-A)$ term in the gravity formula in spite of the equator being circular. This is however not possible, because the potential of a heterogeneous body according to equation $(1 \cdot 24)$ is

$$W = \frac{fM}{r}\left\{1 - \frac{3}{2\,Mr^2}\left(\frac{A+B}{2} - C\right)\left(\sin^2\theta - \frac{1}{3}\right)\right.$$
$$\left. + \frac{3}{4\,Mr^2}(B-A)\cos^2\theta\cos 2L\right\} + \frac{fY_3}{r^4} + \ldots + \frac{1}{2}\omega^2 r^2\cos^2\theta.$$

If its outer surface is to be a spheroid, W must become constant for $r = a\,(1 - \epsilon\sin^2\theta)$. This can only happen when the longitude term is zero. If then a heterogeneous spheroid is to be a figure of equilibrium of masses within it, its internal masses must be so constituted that the equatorial moments of inertia are equal.

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* Darwin, Scientific Papers, **2**, 481–84.

† M. N. R. A. S. Geoph. Suppl., **3**, 1932, 30; **2**, 1932, 60.

‡ Meinesz, Gravity Expeditions at Sea, **2**, 1923–32, 54.

Thus, while one cannot cavil at the longitude term from *a priori* considerations, there is no denying the fact that it has not been strongly determined, because the data are confined to only a very limited portion of the globe. Heiskanen derived his formula (7) of para 2 from 656 gravity stations in Europe, Africa, America and Asia, and found that the introduction of a longitude term [as in formula (8)] decreased the sum of the squares of the anomalies. In 1928*, with more data at his disposal (841 stations including 137 sea stations of V. Meinesz) he obtained formula (9). His 1938 formula comprises about double the above number of stations.† A comparison of the various formulæ reveals that the amplitude of the longitude term is changed from 19×10^{-6} to 28×10^{-6}, and the position of the major axis has a range of 43°. In view of the above, it is not unreasonable to surmise that when homogeneous gravity data become available over the whole globe, we might get an entirely different value for this term.

Jeffreys‡ argues that only two harmonics $\cos^2\theta \cos 2L$ and $\cos^2\theta \sin 2L$ of the second degree have been used for analysing the observed gravity anomalies. The Laplace's function Y_4 contains the terms

$$\frac{15}{2} \cos^2\theta \,(7 \sin^2\theta - 1) \times \begin{cases} \cos 2L \\ \sin 2L. \end{cases}$$

These terms give the same kind of variation in the equatorial ellipticity as the second order terms $3 \cos^2\theta \cos 2L$, $3 \cos^2\theta \sin 2L$. If these fourth order harmonics are present and are not separated by an analysis over different latitudes θ, they will affect the estimated values of the second harmonic. He advocates developing the expression for gravity in spherical harmonics up to terms of the fourth order, and then applying it to deduce the figure of the earth. But Y_6, Y_8, etc. also contain terms that contribute to equatorial ellipticity. The above will therefore lead to useful results only if Y_2, Y_4, Y_6, etc. are in rapidly descending order of magnitude. It is hard to say whether this would actually be the case. The leading terms represent inequalities of wide extent, and their numerical values may not necessarily be greater than those of the succeeding terms.

6. International gravity formula. — The formula $\gamma_0 = 978 \cdot 049\,(1 + 52884 \times 10^{-7} \sin^2\phi - 59 \times 10^{-7} \sin^2 2\phi)$ was adopted at the meeting of the International Association of Geodesy in 1930 at Stockholm as the International gravity formula. The significance of its various terms and the method of their derivation have been discussed in para 3. The coefficient of the $\sin^2\phi$ term corresponds to the value of $\epsilon = \frac{1}{297}$. This value has been obtained from deflection data in U.S., which is only 1·6% of the area of the whole earth. The main consideration in the adoption of this formula was to ensure uniformity in the expression of gravity anomalies in different countries. It is obvious that the complicated gravity

* Heiskanen, Gerl. Beit. z. Geoph. **19**, 1928, 356–77.

† Heiskanen, Publications of the Isostatic Institute of the International Association of Geodesy, No. 1, 1938.

‡ Jeffreys, Gerl. Beit. z. Geoph. **36**, 1932, 210.

distribution on the globe would require more elaborate formula for its adequate representation. For example, the gravity anomalies in India based on this formula are shown in chart xi, Survey of India Geodetic Report 1938. The negative values are strikingly predominant, indicating that this formula does not fit India well. Anomalies with respect to Helmert's 1901 formula are shown in chart x, Survey of India Geodetic Report 1938, and are obviously more balanced.

In East Africa also, Bullard's* work shows that the Hayford anomalies with respect to the International spheroid are predominantly negative. A suitable longitude term can be introduced to give a positive correction to Δg's, so that the preponderance of negative values is decreased. As it happens, the longitude term found by Heiskanen [gravity formula (10)] gives such a positive correction to the anomalies in India. Its value at some points is tabulated below:

Point	$978 \cdot 052 \times 28 \times 10^{-6} \cos(2L + 50^\circ)\cos^2\phi$ mgals
$L = 64^\circ, \phi = 26^\circ$	−22
$L = 78$ $\phi = 32$	−18
$\phi = 28$	−19
$\phi = 16$	−23
$\phi = 12$	−24
$L = 92, \phi = 26$	−13
$\phi = 12$	−15
$L = 100, \phi = 28$	− 7
$\phi = 16$	− 9
$\phi = 12$	− 9

If the gravity anomalies are reckoned with respect to the International formula, the longitude terms of the formulæ (2, 3, 8, 9 and 10) all give a positive correction of about 18 mgals to the anomalies in India. It is obvious then that so far as India is concerned, the triaxial formulæ will give the same anomalies as Helmert's 1901 formula, since the value of G_e in the latter formula is about 20 mgals less.

One reason for introducing more harmonics in the gravity formula has been already given in the last para. We shall discuss this further in chapter v, para 7. With the present gravity material however, it is not possible to improve on the International formula with any measure of certainty. This can be seen from the fact that one obtains widely different results for the longitude term according to the number and location of the gravity stations used.

7. Summary.—Gravity observations and arc measurements show that a triaxial ellipsoid fits the geoid better than a spheroid does, but the ellipticity of the geoidal equator is not proved indisputably. This is due to the fact that the data on which the above fit is

* M.N.R.A.S. Geoph. Suppl., 4, 1937, 107.

based are too scanty, being derived from only a very limited portion of the globe. The International gravity formula is by no means a very good fit to the actual observed values of gravity, as can be seen by the large gravity anomalies in India, Gulf of Mexico, Caribbean Sea and the East Indies. It shows that more gravity observations are needed and more harmonics should be introduced in the gravity formula. Until this is done the question of the ellipticity of the equator will remain open.

It is to be remarked that the gravity values used in deriving the best gravity formula should be in terms of the same or well-connected base stations. If for example all American values were smaller or greater than European, an L-term would obviously appear.

CHAPTER III

FIGURES OF EQUILIBRIUM OF A ROTATING EARTH, AND ELLIPTICITIES OF STRATA OF EQUAL DENSITY INSIDE THE EARTH

1. Level surface of a homogeneous rotating fluid.— In the preceding chapter we have found the ellipticity of the level spheroid from the gravity values on it on the assumption that the bounding surface of the rotating mass is nearly spherical and is at the same time an equipotential. We shall now see how we can determine the ellipticity when the internal law of density is known.

If we imagine the earth to be a fluid, elementary hydrostatics enables us to write down the conditions for its equilibrium. The form of its free surface cannot however be found in general. But useful results can be obtained by assuming a form for the free surface and then seeing whether it is a possible form of equilibrium or not. The theory is dealt with in the usual text-books. We will enumerate here some salient points.

The usual condition of equilibrium of an element of a fluid mass rotating with angular velocity ω is

$$dp=\rho\ [(X+\omega^2x)\ dx+\ (Y+\omega^2y)\ dy+Z\,dz], \qquad \ldots \quad (3\cdot1)$$

where dp is the resultant pressure on the element, and $X+\omega^2x$, $Y+\omega^2y$, Z are the components of the resulting force.

If the form of the outer surface of this fluid (which is an equipotential) be assumed to be the spheroid $\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1$, we must have

$$\frac{X+\omega^2x}{x/a^2}=\frac{Y+\omega^2y}{y/b^2}=\frac{Z}{z/c^2}. \qquad \ldots \quad (3\cdot2)$$

X, Y, Z are the components of the force due to a static spheroid. Assuming the spheroid to be of uniform density ρ and substituting the values of X, Y, Z (see chap. I, para 4) in $(3\cdot2)$, we obtain the following conditions for equilibrium:

If $\frac{\omega^2}{2\pi f\rho} > 0\cdot2247$, an oblate spheroid is not a possible form.

If $\frac{\omega^2}{2\pi f\rho} < 0\cdot2247$, two spheroidal forms are possible. In the above, f denotes the gravitational constant; its numerical value is taken as $6\cdot6\times10^{-8}$ cm.^{3}/gm. sec^2. Taking $\rho=5\cdot5$ gm./cm.^{3}, the limiting value

$$\frac{\omega^2}{2\pi f\rho}=0\cdot2247 \text{ gives } \frac{2\pi}{\omega}=2\tfrac{1}{2} \text{ hours.}$$

The shortest period, therefore, in which a homogeneous fluid having the same mean density as the earth can rotate uniformly in the form of a spheroid is $2\frac{1}{2}$ hours.

For the ρ and ω appertaining to the earth, two spheroidal forms are possible. The larger spheroid has rather a big ellipticity and is of no interest from our point of view. The ellipticity of the other spheroid is given by

$$\epsilon = \frac{15\,\omega^2}{16\pi f\rho} \doteq \frac{1}{232}.$$

Modern observations show the ellipticity of earth to be in the neighbourhood of $\frac{1}{297}$. The large difference is due to the fact that the earth is not a homogeneous fluid mass.

It is of interest also to mention the results obtained regarding the equilibrium of a fluid in the form of a triaxial ellipsoid. Jacobi* proved that an ellipsoid with three axes, the smallest of which coincides with the axis of rotation, is a possible form of equilibrium, subject to a certain limitation of the ellipticities. If ϵ, η denote the meridional and equatorial ellipticities of a Jacobian ellipsoid, then $\epsilon = \frac{1}{2}\lambda^2$, $\eta = \frac{1}{2}\lambda'^2$, where either λ or $\lambda' > 1$. In the case of the earth, we have roughly $\epsilon = \frac{1}{297}$, $\eta = 0\,(\epsilon^2)$. A homogeneous triaxial ellipsoid having the same ellipticities as the earth is therefore not a possible form of equilibrium. For further information on Jacobian ellipsoid, reference may be made to Darwin's † work.

The above considerations are of a theoretical nature in that they apply to a homogeneous rotating fluid. The earth, we know, is definitely non-homogeneous. The next step forward is due to Clairaut‡ who published his book on the figure of the earth about half a century after the third book of Newton's *Principia*. We will now give an account of his theory, and its extension by Darwin and de Sitter.

2. Clairaut's theory.—In Clairaut's theory, the earth is assumed to be heterogeneous, but such that it is built hydrostatically. This implies that the surfaces of equal density are equipotentials. Strictly speaking, therefore, the theory is only applicable below the depth of compensation. It gives the ellipticities of surfaces of equal density inside the earth, as well as the ellipticity of the geoid which is the boundary surface, provided the law of variation of density with depth is known. The following is a brief proof of the well-known Clairaut's differential equation.

Since the earth is in hydrostatic equilibrium, there is no shearing stress inside. It can be seen easily § that such an earth can differ from a sphere by only a second order harmonic. The level surfaces are therefore of the form $r = k\,(1 + Y_2)$.

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* Clarke, Geodesy, 78.
† Darwin, Scientific Papers, **3**, 1910, 119.
‡ Clairaut, Théorie de la figure de la Terre, 1743.
§ Pratt, The Figure of the Earth, 1865, 78.

The potential of a heterogeneous body $r = k(1 + \Sigma Y_n)$ at an internal point* (k_1, θ', ϕ') is

$$U_i = \frac{4}{3}\pi f \int_0^{k_1} \rho' \frac{\delta}{\delta k'} \left\{ \frac{k'^3}{r} + \frac{k'^4 Y_1'}{r^2} + \frac{3}{5}\frac{k'^5 Y_2'}{r^3} + \ldots\ldots \right\} dk'$$
$$+ \frac{4}{3}\pi f \int_{k_1}^{k} \rho' \frac{\delta}{\delta k'} \left\{ \frac{3}{2}k'^2 + k' r Y_1' + \frac{3}{5} r^2 Y_2' + \ldots\ldots \right\} dk',$$

where k is the mean radius of the outermost surface, and k_1 is the mean radius of the stratum of equal density through the point in question. ρ' denotes the value of density at the level k', and the Y''s are the same functions of θ' and ϕ' as the Y's are of θ and ϕ. Taking the equation of our stratum to be $r = k_1(1 - \frac{2}{3}\epsilon_1 P_2)$ and the density at this level to be ρ, the condition that U_i is constant on it gives the equation

$$-\frac{\epsilon_1}{k_1}\int_0^{k_1} \rho' k'^2 dk' + \frac{1}{5k_1^3}\int_0^{k_1} \rho' \frac{d}{dk'}(k'^5 \epsilon')\, dk' + \frac{k_1^2}{5}\int_{k_1}^{k} \rho' \frac{d\epsilon'}{dk'} dk'$$
$$= -\frac{1}{8\pi f}\omega^2 k_1^2. \quad \ldots (3\cdot3)$$

ϵ' denotes the ellipticity of the level surface having the mean radius k'. In the derivation of the above equation, quantities of $O(\epsilon_1^2)$ have been neglected. Differentiating this twice with respect to k_1 and simplifying, we have

$$\frac{d^2\epsilon_1}{dk_1^2} + \frac{6\rho k_1^2}{S(k_1)}\frac{d\epsilon_1}{dk_1} - \left(1 - \frac{\rho k_1^3}{S(k_1)}\right)\frac{6\epsilon_1}{k_1^2} = 0, \quad \ldots (3\cdot4)$$

where $S(k_1) = 3\int_0^{k_1} \rho' k'^2 dk'$, and k_1 is the mean radius of the level surface whose ellipticity is ϵ_1.

This differential equation can also be obtained by utilising the condition that the gravity vector at any point of a level surface is along the normal.

The exact manner of distribution of density inside the crust is not known, but assuming that it increases as we go towards the centre of the earth, it can be seen† from equation $(3\cdot4)$ that ϵ decreases as we go downwards. In other words the level surfaces become more and more spherical as we approach the centre.

To make equation $(3\cdot4)$ integrable, several transformations have been used. We will mention only the elegant transformation of Radau‡ (1885), which gives some very important results. He introduces a variable η § defined by

$$\eta = \frac{k_1}{\epsilon_1}\frac{d\epsilon_1}{dk_1}, \quad \ldots (3\cdot5)$$

and obtains

$$\frac{C}{Mk_1^2} = \frac{2}{3}\left\{1 - \frac{2}{5}\sqrt{1+\eta}\right\}, \quad \ldots (3\cdot6)$$

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* Routh, Analytical Statics, vol. II, § 297.

† Jeffreys, The Earth, 1929, 211.

‡ Comptes Rendus, **100**, 1885, 972–77.

§ This η should not be confused with the equatorial ellipticity of the geoid.

where M, C are the mass and moment of inertia (about the axis of rotation) respectively of the matter enclosed by the surface. If the density distribution inside the level surface is known, $\frac{C}{Mk_1^2}$ can be computed and from it η obtained by equation (3·6). Knowing η, we can get ϵ_1 at any depth by integrating the differential equation (3·5), viz.

$$\frac{d\epsilon_1}{\epsilon_1} = \eta \frac{dk_1}{k_1}.$$

Assuming the ellipticity of the outside surface to be $0 \cdot 337 \times 10^{-2}$, and inferring the density distribution by a trial and error method from a study of near earthquakes and the surfaces of discontinuity, Bullen* gets the following table for ϵ_1 at various depths d.

d	ρ	$\epsilon_1 \times 10^2$	$\frac{1}{\epsilon_1}$
km.			
0	...	0·337	296·7
100	3·38	0·334	299·4
400	4·08	0·325	307·7
1000	4·52	0·308	324·7
2000	5·02	0·278	359·7
2900	5·47 9·93	0·260	384·6
4000	11·21	0·258	387·6
5000	11·93	0·257	389·1
6370	12·26	0·256	390·6

For the geoid the ellipticity ϵ can be obtained from a knowledge of the precessional constant $\frac{C-A}{C}$ by means of the equation

$$\frac{C-A}{C} = \frac{\epsilon - \frac{1}{2}m''}{1 - \frac{2}{5}\sqrt{1+\eta}}, \qquad \ldots \quad (3 \cdot 7)$$

where

$$\eta = \frac{5}{2}\frac{m''}{\epsilon} - 2 \text{ and } m'' = \frac{\omega^2 k^3}{fM}. \qquad \ldots \quad (3 \cdot 8)$$

* M. N. R. A. S. Geoph. Suppl. **3**, 1936, 395–401.

In the derivation of Clairaut's equation, terms of $0\ (\epsilon^2)$ were neglected. Darwin* and de Sitter† have extended Clairaut's theory to terms of second order. Their method is identical with that of Helmert, already described in chap. I, para 6. The external potential of a body symmetrical with respect to its axis of rotation may be written as

$$U = \frac{fM}{r}\left\{1 + \frac{K}{2r^2}(1-3\sin^2\theta) + \frac{\omega^2 r^3}{2fM}\cos^2\theta + \frac{D}{r^4}\left(\sin^4\theta - \frac{6}{7}\sin^2\theta + \frac{3}{35}\right)\right\}, \quad \ldots \quad (3\cdot 9)$$

where K and D are two constants characteristic of the body, as we have already seen. This equation holds irrespective of the internal constitution of the body. If the body is an equipotential surface, its equation will be

$$r = a\left[1-\epsilon\sin^2\theta - \left(\frac{3}{8}\epsilon^2 - \frac{1}{4}\chi\right)\sin^2 2\theta\right]. \quad \ldots \quad (3\cdot 10)$$

Gravity on this surface is

$$g = G_e\,(1+\beta\sin^2\theta + \gamma\sin^2 2\theta), \quad \ldots \quad (3\cdot 11)$$

where G_e is given by equation $(1\cdot 41)$,

and
$$\beta = \frac{5}{2}m - \epsilon - \frac{17}{14}\epsilon m - \frac{2}{7}\chi, \quad \ldots \quad (3\cdot 12)$$

$$\gamma = -\frac{7}{8}\epsilon^2 + \frac{15}{8}\epsilon m + \frac{3}{4}\chi. \quad \ldots \quad (3\cdot 13)$$

The constants K and D occurring in the expression $(3\cdot 9)$ for the potential can also be expressed in terms of ϵ and m as

$$\frac{3K}{2a^2} = \epsilon - \frac{1}{2}m + \frac{3}{4}m^2 - \frac{1}{2}\epsilon^2 + \frac{1}{7}\epsilon m - \frac{1}{7}\chi, \quad \ldots \quad (3\cdot 14)$$

$$\frac{D}{a^4} = \frac{7}{2}\epsilon^2 - \frac{5}{2}\epsilon m - \chi. \quad \ldots \quad (3\cdot 15)$$

In chapter II, the values of β, γ deduced by least squares from the available gravity data were used for determining ϵ. Equation $(3\cdot 14)$ affords a more accurate method of determining the ellipticity. The constants K and χ occurring in it are obtained from the following considerations:

$$\frac{3K}{2a^2} = \frac{3}{2}\frac{C-A}{Ma^2} = qH \text{ (say)}, \quad \ldots \quad (3\cdot 16)$$

where
$$H = \frac{C-A}{C}$$

and
$$q = \frac{3}{2}\frac{C}{Ma^2}.$$

* Darwin, Scientific Papers, **3**, 1910, 78–118.

† De Sitter, Proc. of the R. Acad. of Sc. at Amsterdam, **17**, 1915, 1295.

,, Bull. of the Astron. Inst. of the Netherlands **55**, 1924, and **129**, 1927.

H is known by observations. Taking the ratio of the mass of the moon to that of the earth as $\frac{1}{\mu} = 81 \cdot 50 \pm 0 \cdot 07 + \frac{1}{\Delta\mu}$, de Sitter* obtains the expression for H from the Constant of Precession to be

$$H = 0 \cdot 0032774 + 0 \cdot 0000270 \frac{1}{\Delta\mu}.$$

q can only be obtained by making some sort of assumption about the internal constitution of the body. On the hydrostatic hypothesis, de Sitter found

$$q = 0 \cdot 50075 \pm 0 \cdot 00008.$$

He proved that this value will change by an insignificant amount if the earth were isostatic, or even non-isostatic.

It now remains to assign some value to the second order term χ in equation (3·14). Assuming Roche's law of density viz., $\rho_0 = \rho \left[1 - k \left(\frac{a}{a_0} \right)^2 \right]$ where ρ_0 is the mean density of all matter lying inside surface a, Darwin† obtained $\chi = -204 \times 10^{-8}$. By Wiechert's hypothesis, which assumes the earth to consist of a nucleus of radius roughly $\frac{4}{5}$ of that of the earth and density 8·206, and a top layer of density 3·2, he got $\chi = -168 \times 10^{-8}$. The maximum change in the radius vector of the level surface due to these widely different laws of density is about $\frac{1}{2}$ metre. The effect on the ellipticity would be negligible.

Taking $\chi = -204 \times 10^{-8}$ and $m = 0 \cdot 003467753$, de Sitter obtained

$$\frac{1}{\epsilon} = 296 \cdot 92 \pm 0 \cdot 136.$$

In a later paper ‡ he revised his value of H to 0·0032770 and deduced

$$\frac{1}{\epsilon} = 296 \cdot 96 \pm 0 \cdot 10.$$

He claimed this value of ϵ to be much more trustworthy than any derived from geodetic operations or from the motion of the moon.

In the computation of $\frac{3K}{2a^2} = qH$, de Sitter takes the value of H as given by the astronomical observations for precession, and derives q from the assumption of a hydrostatic earth. He assesses the inaccuracy of q due to this assumption which does not correspond with facts. Jeffreys§ gives a method for determining $\frac{3K}{2a^2}$, which is free from the above objection. His method consists in observing the sidereal motions of the moon's node and perigee, and the inclination of the moon's axis. The first two depend on three

* Bull. of the Astron. Inst. of the Netherlands, 55, 1924.
† Darwin, Scientific papers, 3, 1910, 97.
‡ Ibid, 4, 1927, 57–61.
§ M.N.R.A.S. Geoph. Suppl. 4, 1937, 1–13.

constants $\frac{3K}{2a^2}$, J' and K', where

$$\left.\begin{aligned} \frac{3K}{2a^2} &= \frac{3}{2}\cdot\frac{2C-A-B}{Ma^2} \\ J' &= \frac{3}{2}\cdot\frac{2C'-A'-B'}{2M'a'^2} \\ K' &= \frac{3}{2}\cdot\frac{B'-A'}{M'a'^2}. \end{aligned}\right\} \quad \ldots \quad (3\cdot17)$$

The accented letters in these expressions refer to the moon. The inclination of the moon's axis depends on J' and K'. Observations of the above three motions enable us to solve for $\frac{3K}{2a^2}$, J' and K'.

Jeffreys obtains

$$\frac{3K}{2a^2} = 16{,}453 \times 10^{-7} \pm 65 \times 10^{-7}, \quad \ldots \quad (3\cdot18)$$

which corresponds to

$$\frac{1}{\epsilon} = 296\cdot38 \pm 0\cdot51 \quad \ldots \quad (3\cdot19)$$

and is regarded as the best determination of ϵ at the present time.

Also according to Jeffreys

$$q = 0\cdot5017 \pm 0\cdot0018$$

and according to de Sitter

$$q = 0\cdot5007 \pm 0\cdot00008 \quad \ldots \quad (3\cdot20)$$

Jeffreys' g-formula corresponding to ϵ as given by (3·19) is

$$\gamma_0 = 978\cdot051 \left\{ 1 + (5282 \pm 6) \times 10^{-6} \sin^2\phi - 7 \times 10^{-6} \sin^2 2\phi \right\},$$

and de Sitter's formula is

$$\gamma_0 = 978\cdot052 \left\{ 1 + (52884 \pm 11) \times 10^{-7} \sin^2\phi - 75 \times 10^{-7} \sin^2 2\phi \right\}.$$

3. Summary.—A static homogeneous oblate spheroid and an ellipsoid with unequal axes cannot be level surfaces of their own attraction. A homogeneous rotating fluid in the form of a triaxial ellipsoid having the same axes and rotational velocity as the earth cannot also be a surface of equilibrium. An oblate spheroid is however a possible form of equilibrium of such a fluid, the ellipticity depending on the density and the rotational velocity. The ellipticity of the earth deduced on the basis of a homogeneous rotating fluid is about 20 % greater than that indicated by other considerations.

If the oblate spheroid is heterogeneous and is a surface of equilibrium, its external field can be determined as in chap. I, para 6 without a knowledge of the internal mass distribution. Conversely the ellipticity of such a spheroid can be determined from gravity formulæ deduced by least squares. Alternatively, ϵ can be derived by Clairaut's, Darwin's and de Sitter's theory. Clairaut's method utilises the expression for the internal potential, and is applicable to a hydrostatic level surface. It only takes into account terms of the first order in ellipticity, and its application to the actual earth is limited to depths below 40 km. or so. Darwin and de Sitter have developed Clairaut's theory to terms of order ϵ^2 starting from an expression for the external potential without making any assumption about the internal density distribution. In determining the ellipticity they had to compute a constant on the assumption of hydrostatic stress in the interior. This assumption does not accord with facts. Jeffreys has got over this difficulty by inferring this constant from the moon's motion without reference to any hypothesis about the internal state of the earth.

CHAHTER IV

GRAVITY ANOMALIES AS A MEASURE OF SUBTERRANEAN INEQUALITIES OF DENSITY

1. Compensation.—It is now an accepted fact that the larger features of visible topography are compensated in some form or other. In 1854, Pratt published a paper in the Philosophical Transactions of the Royal Society, in which he calculated the plumb-line deflections due to the Himālayas at three stations (Kaliāna, Kaliānpur and Dāmargīda) of the Great Arc series of India. He found these deflections to be greater in amount than their observed values. This led him to formulate his theory * of compensation, namely that the irregularities of mountain surfaces have arisen from the vertical expansion of the earth's crust from depths below. In this way the surface features get underlain by masses of deficient density.

Hayford † in 1912 gave a practical shape to Pratt's theory of compensation and published tables, by which the effect of topography and compensation on the value of gravity at a station could be computed. He assumed that the total mass in every unit vertical column (whether under the oceans or the continents) down to a certain surface called the surface of compensation, is the same. Each column is supposed to be in independent equilibrium. The Hayfordian hypothesis cannot, however, be mechanically true, as it assumes point to point compensation, which implies that the earth's crust offers no resistance to deformation. Geologists have always regarded it as a mathematical abstraction, but we shall see later that it can give very useful results.

Airy ‡ propounded a hypothesis in 1855 that mountains and plateaux have roots below them penetrating into the denser substratum, the whole block floating in hydrostatic equilibrium. This hypothesis accords approximately with modern conceptions of the constitution of the earth and has found more favour than Hayford's. Heiskanen § has brought out tables for this hypothesis of compensation. He assumes the thickness of the crust corresponding to zero elevation of a region to be 40, 60, 80 and 100km., and the difference of density between the crust and the magma in which it is floating to be 0·6 gm./cm.^{3}.

It is universally agreed now that the compensation cannot be local in nature and that some form of regional compensation should be made the basis from which gravity anomalies should be

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* Phil. Trans. of the Royal Soc. of London, 1859, 745.

† Hayford and Bowie. The Effect of Topography and Isostatic Compensation upon the Intensity of Gravity, 1912.

‡ Phil. Trans. of the Royal Soc. of London. **145**, 1855.

§ Bull. Geod. 1931, p. 110.

reckoned. V. Meinesz has produced tables based on the idea that for each topographic feature *dm* there is a corresponding compensation at a depth of 30km. extending laterally to radius *R*. His tables take into account values of *R* ranging from 0 to 232·40km. As another variety of regional compensation of topographic inequalities, V. Meinesz*, assuming the earth's crust to behave like an infinite elastic plate of constant thickness 25km. floating over a magma, whose density is greater than that of the crust by 0·63, has produced tables for the following cases:—

(*i*) Crust is of constant density, and compensation is concentrated at the junction of the crust with the magma.

(*ii*) Compensation is uniformly distributed throughout the 25 km. depth.

Seismological evidence shows that the normal structure of the earth's crust is by no means homogeneous, but consists of three layers possessing different physical properties. The interfaces of these layers, according to Jeffreys†, are about 10km. and 30km. below sea-level, the discontinuity of densities at these layers being about 0·2 and 0·5 gm./cm.^{3} respectively. Modern theories assume that compensation is confined to the interfaces of these layers. In view of this, not much interest attaches to the controversy which raged at one time as to whether Hayford's or Airy's hypothesis accorded better with the observed gravity anomalies.

In the early days, all discussions on gravity were based on Hayford anomalies, since tables‡ on his hypothesis only were available. These tables hold only for perfect compensation, the depth of compensation being assumed to be 113·7 km. and the mass of compensation being taken to be equal to the corresponding topography. More general tables have now been brought out in Italy §, which can be used both for Pratts' and Airy's types of compensation for any reasonable depth of compensation and thickness of the crust. Also, great strides have since been made in producing tables on other hypotheses, and the Isostatic Institute of the International Union of Geodesy at Helsinki has computed the anomalies on as many as twenty-one different hypotheses and in its publication No 5, 1939, results have been published for 3758 gravity stations. More work is, however, needed in this direction, as there are some useful reductions for which no tables have been worked out; for example the inversion reduction, which possesses the property, that it gets rid of the protruding masses above the geoid in such a way that the natural geoid continues to remain the level surface of the new mass distribution.

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*Bull. Geod. 1931, No. 29.

† Jeffreys. "The Earth", chap. VI. A more recent discussion on this subject is given in M. N. R. A. S. Geoph. Suppl. **4**, 1937, 210.

‡ Hayford and Bowie. The Effect of Topography and Isostatic Compensation upon the Intensity of Gravity, 1912.

§ Cassinis, Dore, and Ballarin ; Tavole foundamentali per la riduzione dei valori osservati della gravitá. Pavia, R. Commissione geodetica italiana, nuova serie, No. 13, **27**, 1937.

2. Gravity and geology.—It has been found that no matter what theory of compensation is adopted as a working hypothesis, there are certain regions where considerable gravity anomalies persist. An important use of the gravity anomalies is that they give a clue to these disturbed areas. As examples of such areas may be mentioned Peninsular India, Ferghana Basin, Japan, Dutch East Indies, the Caucasus, Carribean Sea, the great ocean deeps, the oceanic islands and the African Rift Valleys.

It will not be out of place to give a brief account showing how the gravity anomalies have helped to elucidate the geological history of these regions, and have thrown light on the folding lines of the crust. It might be added that the shape of the geoid deduced from plumb-line deflections can give a valuable confirmation of the results deduced from gravity anomalies. These deflections are integrated along the meridians and parallels, and the separation of the geoid from its reference spheroid is determined. If we do the same with Hayford residuals, we get the compensated geoid which is a level surface of the anomalies from Hayford's hypothesis. The undulations of these geoids yield very valuable information about the nature of compensation in an area.

The Hidden Range.—In India, from a discussion of the plumb-line deflections, Burrard* came to the conclusion as early as in 1901 that there were important sub-crustal features which greatly modified the effect of the Himālayas. He postulated the existence of a subterranean chain of rocks in Central India, running east and west, which caused the plumb-line deflections on either side of this area to be in opposite directions. This "Hidden Range" has been clearly brought out by modern work based on the deflections of plumb-line and gravity. Charts x and xi, Survey of India Geodetic Report 1938, show the gravity anomalies in India with respect to the Helmert and International spheroids respectively. In Chart x, there is a wide belt of positive gravity anomalies running right across India from the Bay of Bengal to Karāchi. Chart xi shows the same feature but not to such a marked degree. This is due to the fact that the International formula makes India a region of predominantly negative gravity anomalies. Bullard found the same thing in East Africa, and it is possible that this formula, while good enough for discussing anomalies of the earth as a whole, is not suitable for application to limited areas.

It is obvious from Chart x, that gravity observations in the Bay of Bengal and Arabian Sea are necessary to delineate the extension of the Hidden Range. Another reason why a knowledge of Δg's in these regions would be welcome is that the geoid in India as deduced from plumb-line deflections shows a difference of about 150 feet at two points on the same parallel of latitude 12° at longitudes 80° and 98°. A corroboration of this extraordinary rise with the help of gravity data would be very interesting.

* Survey of India, Professional Paper No. 5.

A feature of the Hidden Range is that it is flanked on both its north and south sides by areas of defective gravity. To the north, the Indo-Gangetic plain between Agra and Jalpaiguri is a region of low gravity anomalies. This area is filled with light alluvium having a density of about 2·2 gm./cm^3. and at first sight it appears as if this must be the cause of the negative anomalies. A little computation shows, however, that the thickness of sediments required to produce this effect would have to be enormous*. Various theories have been advanced about the origin of this Gangetic trough. It was once thought that it is a V-shaped rift† about ten miles deep, produced by the opening of the crust under tension. This theory has not received general acceptance. Geologists now believe that the Himālayas and the Gangetic trough have both arisen from the waves of tectonic folding from the north. These waves created in front of the rising mountain a depression of the nature of a 'fore-deep'.

Glennie‡ has put forward the crustal warp theory to explain the tectonic features of India. According to this, gravity anomalies are due to deviations from the normal arrangement of the three layers comprising the earth's crust. Fig. 1 shows a downwarp. When the intermediate layer is pressed into the dunite, the latter being plastic is raised up at the edges. Negative gravity anomalies indicate a downwarp, and positive ones an upwarp.

Glennie has suggested that under the Gangetic Plain is the southern margin of the great geosyncline which formed the basin of the Tethys. The formation of this geosyncline involved a deep-seated down-warping of the earth's crust, and the Hidden Range marks the line along which the balancing uprise took place.

Dutch East Indies.—Gravimetric observations of this region were made by V. Meinesz§ in 1928 and 1932. He found a narrow strip of strong negative anomalies differing by about 200 mgals from the neighbouring positive anomalies. As in the case of the Hidden Range this negative belt has no direct connection with topography, as it passes sometimes over submarine islands and sometimes over deeps. It runs parallel to the west coast of Sumatra and has been delineated up to the parallel of 5°. Observations are needed in the Bay of Bengal to show whether this joins up with the defective area to the north of Madras, or with the negative strip near Diamond Island and Bassein.

It is a remarkable feature that this negative strip goes by the side of the Mindanao trough and not directly over it, although this trough has depths of over 8000 metres. The same characteristic is observed in the Nares deep in the Atlantic. Here also the axis of the ridge of negative anomalies is not exactly above the deep, but is

* A parallel case is that of the African rift valleys discussed by Bullard in the Phil. Trans. of the Royal Soc. of London, 10th Aug. 1936. Here again the Δg's are negative, but are not solely due to the light sediments at the top.

† Survey of India, Professional Paper No. 17, p. 15.

‡ Survey of India, Professional Paper No. 27, 1932.

§ **Gravity Expeditions at Sea, 2, 1923–32,** shifted unexpectedly towards a neighbouring island ridge. The above are indications of some phenomena going on inside the earth's crust.

V. Meinesz* offers a physical explanation of these strong negative anomalies by his so-called buckling hypothesis. He considers these areas to be regions of mountain formation. What is happening is, that the land at A and B in Fig. 2 is subjected to compressive forces which produce an upwarp C. Due to the enormous compressive forces the crust gets crushed, and like a floating iceberg has a much greater hump D down in the magma than above sea-level. This sub-crustal hump of lighter material is responsible for the negative anomaly.

The buckling hypothesis thus postulates that the folding of the crust forces light matter into the magma, and in regions which are tectonically active this would give rise to anomalies. This hypothesis is not very different from the crustal warp hypothesis. V. Meinesz† has supplemented the above hypothesis by postulating the existence of forces exerted on the crust by the magma, due to dynamical processes in this sub-crustal layer. The horizontal gradient of temperature at the lower boundary of the crust sets up convection currents, and he finds the effect of these with the help of equations of motion for viscous fluids.

The Atlantic Ocean.—V. Meinesz found the anomalies over a great part of the Atlantic Ocean to be positive. Their mean values by Hayford's, Heiskanen's, and V. Meinesz's hypotheses are $+36$, $+32$ and $+28$ mgals respectively. The existence of this positive anomaly over such a wide region is yet to be explained and will throw much light on the problem of oceanic structure. V. Meinesz has offered some tentative explanations and has discussed the relations of the gravity anomalies with the mid-Atlantic ridge. Such a mid-oceanic ridge is also postulated with a fair degree of certainty by the oceanographers in the Indian Ocean. The process of the formation of such ridges is yet an unsolved problem.

It is to be remarked that conditions in the Atlantic are not similar to those in Netherland East Indies, there being no strip of negative anomalies. V. Meinesz concludes from this that this region is not tectonically active.

Ferghana Basin.—The Pāmir region in middle Asia is of great interest, and has been specially chosen by the Isostatic Institute of the International Association for further study. The free-air gravity anomalies at some of the stations in this area are of the order of -150 mgals. Both the Bouguer and free-air anomalies in this extensive region, covering an area of $(5° \times 4°)$ *i.e.* about 70,000 square miles are generally negative, and they become greatest in the Ferghana basin. Erola ‡ has worked out isostatic anomalies

* Gravity Expeditions at Sea, **2**, 1923-32, 118.

† Ibid, 54.

‡ Publications of the Isostatic Institute of the International Association of Geodesy No. 4, Helsinki. 1938.

for this region on ten different hypotheses. In each case he has correlated Δg with the height of the station, first by omitting stations of the Ferghana Valley and secondly by taking all the stations. Using the criterion that that hypothesis corresponds best to the actual structure of the earth's crust, which gives nearly a zero coefficient of the height term, he gets the thickness of the crust to be 35 km. when all stations are taken into account, and 22 km. when stations of the Ferghana valley are omitted.

D. Muschketov* has suggested that the negative gravity anomalies in Ferghana Valley point to a recent rising of the whole region. These negative anomalies from India to Kasakstan confirm the opinion of the geologists that the mountain system of Pāmir-Alay has a large recent epirogenetic rising.

African Rift Valleys.—Several theories have been advanced by geologists about the origin and history of the African Rift Valleys. To distinguish between the various hypotheses, Bullard† in 1933 carried out a gravity survey in East Africa. He worked out the anomalies on seven different hypotheses, and inferred from them that the African plateau is on the whole in isostatic equilibrium, but that the Rift Valleys are underlain by matter of deficient density. He came to the conclusion that Gregory's theory, that Rifts are caused by tension in the crust followed by fracture, is not true and developed Wayland's suggestion that the Rift is formed by folding and faulting under compression. The light surface matter gets thrust into the magma when the block between the fractures is forced down.

Later in 1934, Horsfield‡ took observations at some more stations in the Tanganyika Territory and found the same close association between the Rift Valleys and the negative gravity anomalies.

The Red Sea had been always regarded as a part of the system of Rift Valleys. V. Meinesz's observations, however, showed gravity to be in excess in this region. Observations on land on both sides have shown that the isostatic anomalies are positive over the Red Sea and its coasts, but these anomalies do not extend much inland. The gravity anomalies thus do not lend support to the view that the Red Sea is a part of the African Rifts. It is underlain by heavy masses, and may not even have been formed at the same time.

Over-compensation in mountainous regions.—A question of some interest is whether there are any mountainous regions of the globe which are over-compensated to such an extent that the geoid is depressed there. With the data available so far the answer to this seems to be in the negative, although there are some mountain stations at which the Hayford anomaly is negative, indicating over-compensation. For instance, to the north of

* Angew. Geoph. vol. v, 1936.
† Phil. Trans. of the Royal Soc. of London, 10th Aug. 1936.
‡ M. N. R. A .S., Geoph. Suppl., Jan. 1937, 94.

Kashmīr, the Hayford gravity anomaly at Depsang is −64 mgals, and at Yārkand −67 mgals. This question has been discussed by the author*, who considered the data in several mountainous regions of the globe and inferred that if the geoidal elevations are taken with respect to the spheroid fitting best the area in question, the geoid follows the topography. Another confirmation of the above result is afforded by gravity measurements in Cyprus by Mace†. Observed gravity is found to be much in excess of what can be expected from topography. Correction for topography and compensation increases the discrepancy between observed and normal gravity. The mountains of Cyprus far from being over-compensated have a great mass of heavy rock beneath them.

The thickness of the earth's crust.—Gravity anomalies can give an indication of T, the thickness of the earth's crust. Heiskanen‡ has computed the anomalies on several hypotheses and has utilized the criterion that the value of T which gives the least values of the anomalies is the best. He concluded that the thickness of the crust corresponding to zero elevation of the ground is 30 km. in West Alps and Norway, and 40 to 50 km. in U.S.A. Taking into account the actual mean height of topography, the values of T work out to about 40 km. in West Alps and Norway, and 50 to 60 km. in U.S.A. Bullard § has estimated the thickness of the crust in East Africa, and Erola ‖ in the neighbourhood of Ferghana basin in middle Asia. In the latter region the thickness of the earth's crust corresponding to zero elevation of the topography appears to be 25 km., and under the neighbouring mountains which are about 3,000 metres high, it is about 40 km. These results are in accord with the evidence afforded by the study of earthquakes, and have led to the gradual crumbling away of the earlier belief that the thickness of the crust is of the order of 1,000 miles. Care, however, is needed in defining the meaning of the earth's crust. Modern theories about the constitution of the earth imply that the crust consists of upper layers floating on a denser substratum which is supposed to be in hydrostatic equilibrium. The thickness of the upper layers is taken to be about 40 km. It might be mentioned, however, that the concept that there is no stress difference below this depth is not rigidly correct. The evidence of gravity anomalies and of deep-focus earthquakes shows that the lower layer is not completely devoid of strength.

3. Definition of gravity anomaly.—Strictly speaking a gravity anomaly should be the difference between observed and theoretical gravity, both being referred to the same surface. Such an anomaly will depend on the normal gravity formula used and on the difference between the actual and assumed mass distributions. In actual practice, however, observed gravity g on the earth is reduced to the geoid while the normal gravity γ_0 refers to the reference

* B. L. Gulatee, Proc. of Imp. Acad. of Sc. Bangalore. Vol. v, March 1937.
† M. N. R. A. S., Geoph. Suppl., June 1939.
‡ Heiskanen, Publications of the Finish Geodetic Institute No. 4, 1924.
§ Bullard, Phil. Trans. of Royal Soc. of London, Aug. 1936, 445–531.
‖ Erola, Publications of Isostatic Institute of International Union of Geodesy, No. 4, 1938.

surface. The anomaly $g-\gamma_0$ is therefore not a true gravity anomaly. It is a conventional anomaly, and is partly due to the difference in elevation between the geoid and its reference surface, and partly to the intervening masses between the two surfaces and to the different mass distribution inside the two surfaces. As an example, consider the usual Hayford's gravity anomaly $(g_c-\gamma_0)$. The observed value of gravity on the earth is reduced to the natural geoid, this being the surface to which the elevations on the ground are referred. Data is generally not available to reduce observed gravity to the level of the reference surface. Since g_c refers to the geoid we see that the usual isostatic anomaly $(g_c-\gamma_0)$ is not a true gravity anomaly. With the help of tables* giving the separation u between the natural and isostatic geoids, we can obtain the reduced value g_c' on the compensated geoid. $(g_c'-\gamma_0)$ contours are drawn and are generally used to indicate areas of mass excess or defects. If, however, the separation N between the compensated geoid and its reference spheroid can amount to $\pm$ 1,000 metres, as some geodesists affirm†, these anomalies will be useless for such a discussion. Even if g_c' be corrected for this N by free-air reduction by the addition of a term $\frac{2gN}{a}$, the position will not be quite satisfactory, as there would still be considerable masses intervening between the geoid and its reference spheroid whose direct effect must be taken into account. The free-air term has been a subject of much controversy as it has been claimed by some that it is responsible for the major part of the gravity anomaly.

A point worth remembering is, that this height correction when applied to the conventional anomaly has a tendency to increase the anomaly algebraically. Experience shows that generally regions of positive gravity values are associated with an elevated geoid, and of negative gravity anomalies with a depressed geoid. $(g_c-\gamma_0)$ and $\frac{2gN}{a}$ have therefore the same sign on the whole, and one cannot explain away the conventional anomaly by this so-called indirect effect.

4. Direct and indirect effects.—A mathematical expression for the conventional anomaly‡ can be obtained in the following way:—

Consider a reference spheroid, gravity γ_0 on which is known. Suppose we put on it a coating of surface density σ, whose potential at an external point is S. Let gravity on the level surface, which has the same potential as the reference surface, be g. Our problem is to find an expression for the anomaly $(g-\gamma_0)$.

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* U.S. Department of Commerce, Coast and Geodetic Survey, Sp. Publication No. 199, 1936.

† Ackerl, Zeit. f. Geophys. **9**, 1933. Ledersteger, Zeit f. Geophys. **10**, (1934), p. 246. The general opinion now is that the separation between the two surfaces can amount at the most to 300 feet.

‡ Helmert, Höheren Geodäsie, II, p. 259.

If W is the external potential due to the spheroid and coating, and U the potential due to the spheroid, we have

$$W = U + S.$$

Gravity at a point G on the level surface (Fig. 3) is

$$g = -\left(\frac{\delta W}{\delta n}\right)_{G,}$$

and at the corresponding point P on the spheroid is

$$\gamma_0 = -\left(\frac{\delta U}{\delta n}\right)_{P.}$$

The conventional gravity anomaly is

$$\begin{aligned} g - \gamma_0 &= -\left(\frac{\delta W}{\delta n}\right)_G + \left(\frac{\delta U}{\delta n}\right)_P \\ &= -\left[\left(\frac{\delta U}{\delta n}\right)_G + \left(\frac{\delta S}{\delta n}\right)_G\right] + \left(\frac{\delta U}{\delta n}\right)_P \\ &= -\left[\left(\frac{\delta U}{\delta n}\right)_P - N\left(\frac{\delta \gamma_0}{\delta n}\right)_P + \ldots + \left(\frac{\delta S}{\delta n}\right)_G\right] + \left(\frac{\delta U}{\delta n}\right)_P \\ &\doteqdot -\left[\left(\frac{\delta S}{\delta n}\right)_G - N\left(\frac{\delta \gamma_0}{\delta n}\right)_P\right]. \end{aligned}$$

We make the approximation now, that

$$\left(\frac{\delta S}{\delta n}\right)_{\text{geoid}} = \left(\frac{\delta S}{\delta n}\right)_{\text{spheroid}} = \frac{\delta S}{\delta r},$$

where δr denotes differentiation along the radius vector.

To see the justification for this, suppose the equations of the geoid and its reference spheroid are

$$r_g = a\left[1 - \epsilon_1 f_1(\theta, \psi) - \epsilon_2 f_2(\theta, \psi)\right] \qquad \ldots \quad (4\cdot1)$$

and $$r_s = a\left[1 - \epsilon_1 f_1(\theta, \psi)\right], \qquad \ldots \quad (4\cdot2)$$

where θ, ψ denote the angular co-ordinates of a point. Experience shows that the geoid and its reference spheroid can differ by 200 or 300 feet and not very much more. Hence $a\epsilon_2$ can at the most amount to 300 feet, i.e. $\epsilon_2 = \dfrac{1}{6 \times 10^4} = O(\epsilon^2)$, ϵ_1 being of $O(\epsilon)$.

The angle μ between the radius vector and the normal at a point of the surface (4 · 1) is

$$\tan \mu \doteqdot \mu = \frac{\left\{\sin^2\theta\,(\epsilon_1 f_{1\theta} + \epsilon_2 f_{2\theta})^2 + (\epsilon_1 f_{1\psi} + \epsilon_2 f_{2\psi})^2\right\}^{\frac{1}{2}}}{\sin\theta\left[1 + \epsilon_1 f_1 + \epsilon_2 f_2\right]},$$

where the suffixes θ, ψ denote differentiations with respect to θ and ψ respectively. The angle χ between the normals of the geoid and spheroid is of $O(\epsilon_1 \epsilon_2)$. The error made in taking

$$\left(\frac{\delta S}{\delta n}\right)_{\text{geoid}} = \left(\frac{\delta S}{\delta n}\right)_{\text{spheroid}} \text{ is}$$

$$\frac{\delta S}{\delta n}\left(1 - \cos\chi\right) = O\left(\epsilon_1^2 \epsilon_2^2 \frac{\delta S}{\delta n}\right) = O\left(\epsilon^6 \frac{\delta S}{\delta n}\right),$$ which is negligible.

The error involved in assuming

$\left(\frac{\delta S}{\delta n}\right)_{\text{spheroid}} = \left(\frac{\delta S}{\delta n}\right)_{\text{sphere}}$ is of order $\frac{\delta S}{\delta n} \cdot \mu^2 = O\left(\epsilon^2 \frac{\delta S}{\delta n}\right)$,

which again is negligible.

Also to an accuracy of O ($g\epsilon^3$), we can take

$$N\frac{\delta \gamma_0}{\delta n} = -\frac{2gN}{a}.$$

We finally get

$$g - \gamma_0 = -\left(\frac{\delta S}{\delta r} + \frac{2S}{r}\right)_{r=a,} \qquad \ldots \quad (4\cdot 3)$$

where r denotes the radius vector of a sphere of radius a.

To the above order of approximation, therefore, the difference between the values of gravity on the geoid and its reference surface can be deduced by assuming the coating to be on a sphere. The external and internal potentials due to this coating of skin density $\sigma = \rho\, \Sigma\, H_n$ on a sphere of radius a are

$$\left.\begin{aligned} S_e &= 4\,\pi f a \rho\, \Sigma\, \frac{1}{2n+1}\left(\frac{a}{r}\right)^{n+1} H_n, \\ S_i &= 4\,\pi f a \rho\, \Sigma\, \frac{1}{2n+1}\left(\frac{r}{a}\right)^{n} H_n. \end{aligned}\right\} \qquad \ldots (4\cdot 4)$$

Obviously, for $r = a$ we have

$$\frac{\delta\, S_e}{\delta\, r} + \frac{S_e}{2r} = -2\pi f \rho\, \Sigma\, H_n = -\,2\pi f \sigma$$

Substituting in equation (4 · 3) we have

$$\begin{aligned} g - \gamma_0 &= \left(-\frac{2S}{r} + \frac{S}{2\,r} + 2\pi f \sigma\right)_{r=a} \\ &= \left(-\frac{3S}{2\,r} + 2\pi f \sigma\right)_{r=a} \qquad \ldots (4\cdot 5) \\ &= +\,\Delta_1 g + \Delta_2 g, \end{aligned}$$

where $\Delta_1 g$ is the direct effect of the coating and $\Delta_2 g$ is the indirect effect due to the difference in level between the two surfaces. Equation (4 · 5) can also be expressed as an integral equation in σ, viz.

$2\pi f \sigma - \frac{3}{2a}\int \frac{\sigma d\omega}{r} = \Delta g$, the integration being on a sphere of radius a. Idelson* has indicated a method of solving this with the help of spherical harmonic functions.

The value of the potential on the geoid and its level surface being the same, we have

$$\left.\begin{aligned} S &= Ng \\ \Delta_1 g &= 2\pi f \sigma + \frac{Ng}{2\,a} \\ \Delta_2 g &= -\,\frac{2Ng}{a} \end{aligned}\right\} \qquad \ldots (4\cdot 6)$$

* Gerl. Beit. **40**, 1933, 24.

$\Delta_1 g$ denotes the attraction of the coating, and is made up of two parts; $2\pi f\sigma = \frac{3gD}{4a}$ is the attraction of the near portions and $\frac{Ng}{2a}$ is the effect of remote portions. D is defined by the relation $\sigma = \rho D$, where ρ is the normal density of the earth's crust. If the coating $\sigma = \rho \Sigma H_n = \rho D$ is homogeneous, i.e. if D is constant, the near and remote portions exert equal effects, and we have $N = \frac{3}{2}D$. The mass of the coating is $m = 4\pi a^2 \sigma = 4\pi a^2 D\rho = \frac{8}{3}\pi a^2 N\rho$. If H_0 is zero, the coating becomes massless, and the effect of its remote portions may be considered to be negligible. The indirect effect can also be neglected, provided the separation N between the two surfaces is much less than D. In this case, we obtain the very convenient result that the coating is $\frac{\Delta g}{2\pi}$, and a good estimate of the anomalous masses can be obtained from the gravity anomalies by the use of the simple infinite formula. This would not of course hold for an extensive area, in which N varies within wide limits. At first sight, we have obtained here an apparent contradiction with Green's equivalent stratum, which says that matter M_1 inside an equipotential surface can be replaced by a skin density $\sigma = \frac{\Delta g}{4\pi}$ on the surface. The explanation of the discrepancy lies in the fact that the mass of Green's coating $\frac{\Delta g}{4\pi}$ is not zero but M_1. The effect of near portions is therefore comparable with that of remote ones. We have taken a massless coating, the effect of remote portions of which is negligible. The near portions only are responsible for producing Δg and we can use the infinite plane formula.

The case usually met with is that a series of gravity observations are carried out over a limited area, and from them the masses causing the gravity anomalies are inferred. For solving such problems, it is of great importance to know the disturbance of gravity due to standard forms of anomalous masses. The indirect effect is of secondary importance.

For a study of the mass distribution in an extensive region the indirect effect has to be taken into account. Convenient tables are now available* from which this can be easily obtained. With the help of these tables, the Isostatic Institute of the International Geodetic Association has calculated† the indirect effect of the Hayford zones 1 to 7 at different places on the earth.

5. Subterranean mass anomalies.—For an area covered with a sufficiently dense network of gravity stations, one can deduce valuable information about the inequalities of mass (as

* Tables for determining the form of the geoid, and its indirect effect on gravity. U. S. C. & G. S., Special publication No. 199.

† Bull. Geod, **60**, 1938, 409.

reckoned from an assumed standard distribution) and about the nature of equilibrium from the gravity anomalies. It must however be borne in mind that this problem has no unique solution. So far as the gravitational effects are concerned, any mass can be replaced by an infinity of different mass distributions having the same total mass. The following examples will illustrate this.

A homogeneous sphere, and a point mass at its centre have equivalent effects. Confocal ellipsoids with equal masses have the same external field. The internal masses of an equipotential surface* can be replaced by a skin density $\frac{\Delta g}{4\pi}$, so far as external gravitational effects are concerned. The corresponding theorem for the case when the boundary of the attracting masses is not an equipotential, is that the equipotential of the internal masses is equivalent to that of a skin density $\frac{\Delta g}{4\pi}$ combined with a distribution of doublets of intensity $-\frac{U}{4\pi}$ per unit area with their axes directed normally, U being the internal potential of the masses.

Again from analogy with a well-known electrostatic problem it can be easily shown that the effect of a mass m at C' at depth z below the earth, (Fig. 4) assumed to be a sphere of radius k, is equivalent to a coating of surface density $\sigma = \dfrac{m\left\{b^2-(k-z)^2\right\}}{4\ \pi\ b^2\ r^3}$ on a sphere of radius b; r denotes the distance from the point C', and radius b has to be less than k. From this we infer that a local compensation at depth z can be replaced by a regional compensation at a smaller depth and thus gravity data alone will not suffice to distinguish between these two distributions.

An extension† of this theorem is that a mass distribution in a thick spherical shell bounded by spheres of radii b' and b ($b' < b$) can be replaced by a coating on the surface b, so for as its external effects are concerned. This may be proved as follows.

Potential at P due to an element dm (Fig. 5) at a point (r', θ', ϕ') is $\delta V = \frac{dm}{e}$, e being the distance of P from dm. The potential due to the volume density is

$$V = \int_{b'}^{b}\int_{0}^{2\pi}\int_{-1}^{+1} \frac{\rho\,(r', \theta', \phi')\, r'^2\, d\mu'\, dr'\, d\phi'}{\sqrt{r^2+r'^2-2rr'\cos\zeta}}.$$

$$= \int_{b'}^{b}\int_{0}^{2\pi}\int_{-1}^{+1} r'^2 \rho\,(r', \theta', \phi')\left[\frac{1}{r}\ \mathrm{P}_0\,(\cos\zeta) + \frac{r'}{r^2}\ \mathrm{P}_1\,(\cos\zeta) + \frac{r'^2}{r^3}\ \mathrm{P}_2\,(\cos\zeta) + \ldots\right] d\omega \quad \text{for } r > r',$$

* Routh's Statics, 77.
† MacRobert, Spherical Harmonics, 163.

$$\text{and } V = \int_l^b \int_0^{2\pi} \int_{-1}^{+1} r'^2 \rho (r', \theta', \phi') \left[\frac{1}{r'} P_0 (\cos \zeta) + \frac{r}{r'^2} P_1(\cos \zeta) + \frac{r^2}{r'^3} P_2 (\cos \zeta) + \ldots \right] d\omega \text{ for } r < r'.$$

Denoting the external and internal potentials by V_e and V_i, we have

$$V_e = \sum_0^{\infty} \int \int \int \rho (r', \theta', \phi') \frac{r'^{\,n+2}}{r^{n+1}} P_n (\cos \zeta) \, d\mu' \, dr' \, d\phi' \quad \ldots (4 \cdot 7)$$

$$\text{and } V_i = \sum_0^{\infty} \int \int \int \rho (r', \theta', \phi') \frac{r^n}{r'^{\,n-1}} P_n (\cos \zeta) \, d\mu' \, dr' \, d\phi' \quad \ldots (4 \cdot 8)$$

If we know the law of variation of density ρ, the external and internal potentials are known.

For a given value of r', let $\rho (r', \theta', \phi')$ be expanded in the form

$$\rho (r', \theta' \phi') = \Sigma \, u_n (r', \theta', \phi').$$

$$\text{Then } V_e = \Sigma \Sigma \int_{b'}^{b} \int_0^{2\pi} \int_{-1}^{+1} u_n P_n \frac{r'^{\,n+2}}{r^{\,n+1}} d\omega$$

$$= 4 \pi \Sigma \frac{1}{(2n+1)} \frac{1}{r^{n+1}} \int_{b'}^{b} u_n (\theta, \phi, r') \, r'^{\,n+2} \, dr'.$$

The potential due to a coating $\sigma = \Sigma \, v_n$ on a sphere of radius b at an external point r' is

$$V = 4 \pi \Sigma \frac{b^{n+2}}{2n+1} \cdot \frac{1}{r'^{n+1}} v_n (\theta, \phi).$$

The two potentials are equal, if

$$v_n = \int_{b'}^{b} u_n (r', \theta, \phi) \left(\frac{r'}{b} \right)^{n+2} dr'.$$

Knowing u_n, we can determine v_n. It is to be noted that the coating has the same mass as the matter inside the spherical shell.

The above examples amply illustrate the fact that it is not possible to infer the exact distribution of the disturbing masses from the gravity anomalies; the total sum of the disturbing masses can, however, be deduced from them. This can be seen by an application of Gauss' Theorem that if Δg is the attraction due to a system of masses inside a body, $\iint \Delta g dS = 4\pi M$, where the integration is carried over the surface of the body and M is the total sum of the masses. For a plane area, $\iint \Delta g dS = 2\pi M$. The value of this result lies in the fact that if the disturbing masses are at small depths, their effects fade off quickly with the distance. Consequently the integral of the Δg's in a limited disturbed area will approximately give the magnitude of the masses causing them. This result is valuable since the gravity anomalies and undulations of the geoid depend more on the total disturbing masses than on their disposition.

It should, however, be borne in mind that the number of solutions is limited in practice by certain considerations. As an example of this we will show that a set of gravity anomalies can be explained theoretically by placing suitable masses at any given depth. In actual practice, however, if the masses are placed below a certain depth, the extent required is such as to make their existence physically improbable. To see this, suppose the gravity anomalies on a sphere of radius a are expressed in a series of spherical harmonic functions as $\Delta g = \Sigma\, g_n\, Y_n$. We shall see later that these can be explained by a surface density $\sigma = \Sigma\, \sigma_n Y_n$ on a sphere of radius $(a-z)$, where

$$\sigma_n = \frac{g_n}{4\pi f}\left(\frac{2n+1}{n+1}\right)\left(\frac{a}{a-z}\right)^{n+2}. \qquad \dots (4\cdot 9)$$

To find the effect of the depth, consider a series of warps of wave-length 300 miles on a sphere of radius 3960 miles so that

$$\frac{2\pi a}{n} = 300.$$

In equation (4·9) replacing σ_n by an equivalent thickness of H miles of rock of volume density 0·02 gm/cm.^{3}, we see that to produce an anomaly of 0·02 cm./sec.^{2}, we must have $H = 1\cdot 47$ miles for depth $z = 0$, and $H = 1\cdot 53$ miles for $z = 2$ miles. For this small increase of depth, therefore, the increase of H does not amount to much; but when the increase of depth is comparable to the wave-length, a considerable increase of thickness H is required. Thus for $z = 150$ miles, i.e. at a depth of half the wave-length, $H = 39$ miles.

To further illustrate the point, consider the cases of a plane, a spherical and a cylindrical deposit. The attraction of a circular disc of radius a and surface density σ at a point height h, is

$$\Delta g = 2\pi\sigma f\left(1 - \frac{h}{\sqrt{a^2+h^2}}\right) = 2\pi\sigma f F \text{ (say)}.$$

The variation of F with h/a is shown in the following table:

$\frac{h}{a}$	F	$\frac{h}{a}$	F
0·00	1·00	1	0·29
·01	0·99	2	·11
·05	·95	3	·05
·10	·90	10	·005
·60	·49	20	·0013
0·80	0·37	100	0·00005

We see that for a given disc, F decreases but slightly with depth when h/a is small. A particular case, in which the attraction is independent of the depth, is that of the infinite plane corresponding to $h/a = 0$. For large values of h/a, however, F decreases as the inverse square of h/a. In other words, doubling the depth of the plane mass necessitates a fourfold increase of density to produce the same effect.

A spherical mass of radius 10 km. having a density difference of $0 \cdot 1$ gm./cm.^{3} from the surrounding material, will produce a $\Delta g = 0 \cdot 028$ cm./sec.^{2}, when it is tangential to the ground surface, and a $\Delta g = 0 \cdot 003$ cm./sec.^{2}, when its centre is at a depth of 30 km. At a depth of 30 km. or more, therefore, its effect cannot be measured with certainty. If the given gravity anomalies have to be explained by such a mass at a depth of 30 km., we will have to postulate for it a much greater difference of density from the surrounding masses.

A cylindrical mass of radius $\frac{1}{2}$ mile and thickness 1000 feet produces a $\Delta g = 0 \cdot 017$ cm./sec.^{2} at a point on its axis, 100 feet above its upper surface. When placed at a depth of 12,000 feet its effect is only $0 \cdot 00089$ cm./sec.^{2}, i.e. about 20 times less.

The foregoing examples show that it is possible to infer from the gravity anomalies an upper limit to the depth at which the disturbing masses can lie. At greater depths the density inequalities required will be too great to be physically possible.

6. Mass anomalies expressed as a coating.—In determining the difference in arrangement of the masses inside the earth from an assumed standard distribution (like the isostatic), it is best to idealize the earth in such a way that all masses protruding above the geoid are removed. One method of doing this is by the usual isostatic reduction, which removes all the topography external to the geoid, and also its compensation as postulated by Hayford. The equipotential surface of the new mass system, which has the same potential as the geoid ($V = C_0$), is designated as the compensated geoid.

Let R be a uniform spheroid (Fig. 6l) so chosen that the value of potential over it is $U = C_0$, and such that its volume is equal to that of the compensated geoid. The compensated geoid is the equipotential of the matter within the following surfaces: uniform spheroid R, matter A between the compensated geoid and its reference spheroid R, matter B between the compensated and natural geoids, and the anomalies from uniformity. These anomalies may either be deep-seated or close to the surface. Let their effect be equivalent to a skin distribution σ_1 on the spheroid. Hence the compensated geoid is the equipotential of the uniform spheroid + skin density σ_1 + matter $\rho \, (N + N_c)$, where ρ denotes the density of the earth's crust, N_c the height of natural geoid above the compensated geoid, and N the height of the compensated geoid above the spheroid R.

Imagine the mass $\rho \, (N + N_c)$ to be condensed as a skin density on the spheroid, and let $\sigma_1 + \rho \, (N + N_c) = \sigma$. Our new mass distribution, then, is a spheroid R + a skin density σ on it. We will designate by corrected geoid that equipotential surface of this new mass system, which has the same potential as the compensated geoid. The separation N_d of the corrected geoid from the spheroid is due to the effect of the skin density σ. This concept of skin density is very useful for the solution of many problems; if one deals with the three dimensional distribution of mass as found in nature, the corresponding formulæ become unmanageable.

As argued before, so far as the effect of coating is concerned the spheroid may be replaced by a sphere of appropriate radius. The potential δV at an external point due to coating $\sigma = \rho \Sigma Y_n$ on a sphere of radius k is

$$\delta V = 4\pi f \rho k \Sigma \frac{1}{2n+1}\left(\frac{k}{r}\right)^{n+1} Y_n$$
$$= 4\pi f \rho k \Sigma \frac{Y_n}{2n+1} \text{ at } r = k.$$

Hence, assuming the ratio between the crustal and mean density of the earth to be $2 \cdot 07$, we have $N_d = \dfrac{\delta V}{G} = \dfrac{3}{2 \cdot 07} \Sigma \dfrac{Y_n}{2n+1}$.

The attraction of this coating at $r = k$ is

$$\Delta g_r = -\frac{\delta}{\delta r}(\delta V) = 4\pi f \rho \Sigma \frac{n+1}{2n+1} Y_n$$
$$= 2\pi f \rho \Sigma \left(Y_n + \frac{Y_n}{2n+1}\right)$$
$$= 2\pi f \rho \left[\frac{\sigma_1 + \rho(N + N_c)}{\rho} + \frac{2 \cdot 07}{3} N_d\right]. \qquad \ldots \quad (4 \cdot 10)$$

If g_d denotes the value of gravity on the corrected geoid, and γ_o on the spheroid, excluding the attraction of σ, then

$$g_d - \gamma_o = \Delta g_r - \frac{2G}{k} N_d$$
$$= 2\pi f \rho \left[\frac{\sigma_1 + \rho(N + N_c)}{\rho} + \frac{2 \cdot 07}{3} N_d - \frac{8 \cdot 28}{3} N_d\right]$$
$$= 2\pi f \sigma_1 + 2\pi f \rho (N + N_c - 2 \cdot 07 N_d) \qquad \ldots \quad (4 \cdot 11)$$

Let σ_1 be equivalent to a thickness of H feet of rock of normal density, i.e. $\sigma_1 = \rho H$. H represents the mass anomaly measured in feet of rock of normal density, and is given by the expression

$$H = \frac{1}{2\pi f \rho}(g_d - \gamma_o) - (N + N_c - 2 \cdot 07 N_d).$$

Taking $f = 6 \cdot 68 \times 10^{-8}$ cm.^{3}/gm. sec.^{2} and $\rho = 2 \cdot 67$ gm./cm.^{3}, we have H (in feet) $= 29 \cdot 2 \times 10^3 (g_d - \gamma_o) - (N + N_c - 2 \cdot 07 N_d)$. $(4 \cdot 12)$

This formula has to be reduced a bit further, as in practice we do not know the corrected geoid. Let P be a point on the compensated geoid, and P_d a point on the corrected geoid, vertically below or above P. From Fig. 6 we see that

$$g_c = \left(\gamma_o - \frac{2\gamma_o N}{k}\right) + \text{attraction of } \sigma_1 + \text{attraction of matter } \rho(N + N_c) \text{ at } P,$$

and $g_d = \left(\gamma_o - \frac{2\gamma_o N_d}{k}\right)$ + attraction of coating $\left\{\sigma_1 + \rho(N + N_c)\right\}$ at P_d.

Hence $g_c - g_d = \left\{\begin{array}{l}\text{attraction of matter } \rho(N + N_c) \text{ at } P - \\ \text{attraction of coating } \rho(N + N_c) \text{ at } P_d\end{array}\right\}$

$$+ \frac{2\gamma_o}{k}(N_d - N)$$

$$= \frac{2\gamma_o}{k}(N_d - N) + \left(\iiint \frac{dm}{r} - \iint \frac{\sigma_0 dS}{r}\right), \quad \ldots \quad (4\cdot13)$$

where the volume integral extends throughout the space $(N + N_c)$, and $\sigma_0 = \rho(N + N_c)$ is the skin density on the spheroid. The terms in integrals on the right-hand side can be evaluated rigorously with the help of Hayford's reduction tables, provided the undulations $(N + N_c)$ are known all over the globe. As far as our present knowledge goes, $(N + N_c)$ can have a maximum range of 500 feet. It is quite easy to show that except for very uneven terrain, the second term within brackets on the right-hand side of equation (4·13) is negligible. The first term amounts to 0·001 cm./sec.^{2} for $N_d - N = 10$ feet. The average distance between the compensated and corrected geoids is much less than this; hence for all practical purposes, g_c may be put equal to g_d. Also to the order of accuracy to which we are working we may put $N_d = N$ in (4·12). Our final expression for the anomaly then becomes

$$H = 29\cdot2 \times 10^3 (g_c - \gamma_o) - (N_c - 1\cdot07 N). \quad \ldots \quad (4\cdot14)$$

The difficulty in the practical application of this formula is that the spheroid used for computing γ_o is oriented differently to that from which N_c and N are reckoned. This difficulty will remain until we refer our triangulations to an earth spheroid, or until a sufficiently dense mesh of gravity stations is available on the globe, from which N can be computed by Stokes' formula.

In India, N as evidenced from a study of the plumb-line deflections ranges from -20 to $+140$ feet, but most of the change is located in the narrow strip between Mandalay and Mergui. If one neglects this portion, N ranges from -20 to $+40$ feet, i.e. displays a range of 60 feet only. This thickness of matter, even if it be of infinite extent and of as great a density as the normal density of the crust, produces a gravity effect of only 2 mgals. Mass anomalies of this order are not of much interest. If we substitute in equation (4·14) the gravity anomalies in India, we see that H varies from $-2{,}000$ to $+1{,}000$ feet. In particular, the Gangetic plane is an area of underload, the deficiency there being equivalent to a skin density of -500 to $-2{,}000$ feet of rock condensed on the spheroid. These are rather large departures from isostasy, but one cannot argue from this that there is no compensation. If this were so, rigorous topographic reduction should give zero anomalies which is by no means the case. Some sort of compensation has to be postulated.

Instead of assuming the standard earth to be isostatic we might start with the three-layered crust, and infer from Δg's the interactions of the different layers with one another. As an example of fitting observed gravity anomalies by trial and error by assuming warps at the interfaces, may be mentioned the work of Ansel*.

The above discussion holds when the objective is to find the lack of equilibrium of an extensive region or of the earth as a whole from some assumed normal state. We employ the earth spheroid, and it is imperative to use a physically plausible gravity reduction like the regional, because we want to get the absolute value of H.

When, however, the area under investigation is of comparatively small dimensions as in geophysical exploration, the thing of interest is the variation of H and not its absolute magnitude. We can now use the spheroid of best fit to the area in question, and the reduction to be employed need not be mathematically rigorous, since the effect of distant portions is practically constant over this limited area. As examples of such reductions may be mentioned the flat earth Bouguer, V. Meinesz's modified Bouguer, and Glennie's $(g-\gamma_F)$ reduction.

The indirect effect due to the separation between the geoid and the spheroid is of no moment in this case, and it is customary to find the disturbing masses by trial and error. The following table will illustrate the case in point. It gives $\Delta_1 g$, $\Delta_2 g$ at a point in the centre of a spherical cap of radius r and height h.

r km.	100	50	100	50	10
h km.	3	3	1	1	1
$\Delta_1 g$ gals	0·341	0·336	0·115	0·114	0·110
$\Delta_2 g$ gals	0·011	− 0·005	− 0·004	− 0·002	0·000

The indirect effect can manifestly be neglected. We will discuss in the next para the attractions due to different types of attracting masses.

7. Direct effect.—

(*i*) *Infinite plane with a constant surface density* σ. Attraction due to this is

$$\Delta g = 2\pi f \sigma. \quad \dots \quad \dots \quad (4\cdot 15)$$

From this we see that 30 feet of rock of density 2·67 produces $\Delta g = 0\cdot 001$ gal. This is a very useful rule for rough estimations.

(*ii*) *An infinite plane at depth* z *with a surface density* $\sigma = \sigma_0 \cos n x'$ (Fig. 7).

At a point A (x, z) on the earth, the vertical attraction due to an element dx' is

$$2fz \cdot \frac{\sigma \, d x'}{(x'-x)^2 + z^2}. \quad \dots \quad (4\cdot 16)$$

* Angew. Geoph. 6, 1937, 141.

Hence, due to the whole plane,

$$\Delta g\,(x) = 2fz \int_{-\infty}^{+\infty} \frac{\sigma_0 \cos n\,x'\,dx'}{(x'-x)^2+z^2} = 2\,f\,\sigma_0\,e^{-nz}\cos nx . \qquad \dots \quad (4\cdot 17)$$

If we know z, we can get an idea of σ_0 from the known values of $\Delta g\,(x)$.

As an example, suppose we want to explain the g-anomalies on the Hidden Range in India by assuming an upwarp in the crustal layer at depth $z=2$ miles (say). If we take the Hidden Range to be a part of a series of harmonic undulations of wave-length 300 miles, we have

$$n = \frac{2\pi}{300} \text{ and } z = 2.$$

Hence, by equation (4·17),

$$|\,\sigma_0\,| = \frac{\Delta g}{2f}\,e^{\frac{4\pi}{300}} = \frac{\Delta g}{13\cdot 4\times 10^{-8}} \times e^{\frac{4\pi}{300}} = 7\cdot 8\times 10^6\,\Delta g.$$

Let σ_0 be equal to $H\,\Delta\rho_0$ where $\Delta\rho_0 = 0\cdot 2$ gm/cm.^{3}, so that H will be the thickness in cm. of matter of volume density $0\cdot 2$ gm./cm.^{3} Then we have

$$H\,(\text{in cm.}) = \frac{7\cdot 8\times 10^6}{0\cdot 2}\,\Delta g = 39\times 10^6\,\Delta g.$$

Taking the mean anomaly on the top of the Hidden Range to be $0\cdot 02$ cm./sec.^{2}, we get $H = 4\cdot 8$ miles.

(*iii*) *Spherical disc** (Fig. 8). The attraction of a spherical disc of radius k, surface density σ and angular extent θ, at a point at height h above its middle point, is

$$\Delta_1 g = \frac{\pi\,k\,f\,\sigma}{(k+h)^2}\left[\frac{4\,k\,(k+h)\sin^2\frac{\theta}{2} - 2\,h\,k}{\sqrt{h^2+4\,k\,(k+h)\sin^2\frac{\theta}{2}}}\right]_0^\theta \qquad \dots \quad (4\cdot 18)$$

When h is small compared to k, we have

$$\Delta_1 g = 2\,\pi f\sigma\left[\sin\frac{\theta}{2} - \frac{h}{2k}\left(\operatorname{cosec}\frac{\theta}{2} + 3\sin\frac{\theta}{2}\right) + \frac{h}{8k^2}\left(15\sin\frac{\theta}{2} + 9\operatorname{cosec}\frac{\theta}{2}\right) + \left(1 - \frac{2h}{k} + \frac{3h^2}{k^2}\right)\right]. \quad \dots \quad (4\cdot 19)$$

(*iv*) *Spherical coating*†. In sub-head (*iii*) we have considered a uniform spherical coating of limited angular extent θ. We will now deal with the more important case of heterogeneous coatings extending over the whole sphere and will consider in turn the anomalies produced by

(*a*) an uncompensated skin density

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* Helmert, Höheren Geodäsie **2**, 1884, 89.

† Formulæ similar to the ones in this para have been used by Jeffreys in "The Earth", p. 221 and by Stoneley in M.N.R.A.S. Geoph. Suppl. **3**, 1933, 176.

(*b*) a compensated skin density

(*c*) undulations at the interfaces of different layers of the earth's crust at known depths.

(*a*) *Uncompensated skin density.*—Consider a spheroid having the same volume as the earth. The actual earth is, therefore, made up of this spheroid with uncompensated topography of total mass zero superposed on it. An idea of the effects of this mass may be obtained by condensing it as a coating of skin density σ on a sphere having the same volume as the spheroid. Let the level surface $r=a$ be deformed by an amount N on account of the superposition of skin density $\sigma = \Sigma \sigma_n S_n$ on it, S_n being a Laplace's function of order n.

The potential due to this coating is

$$V = \Sigma \frac{4\pi f}{2n+1} \cdot \frac{a^{n+2}}{r^{n+1}} \sigma_n S_n.$$

Hence $$N = \frac{V}{G} = \frac{1}{G} \Sigma \frac{4\pi f a}{2n+1} \sigma_n S_n.$$

The direct effect of the coating at $r=a$ is

$$\Delta g = -\frac{\delta V}{\delta r} = \Sigma\, 4\pi f \frac{n+1}{2n+1} \sigma_n S_n. \qquad \ldots \quad (4\cdot 20)$$

If the indirect effect is also taken into account, the usual conventional anomaly is

$$\Delta g_1 = \Delta g - \frac{2V}{r} = 4\pi f\, \Sigma \frac{n-1}{2n+1} \sigma_n S_n \quad \ldots \quad (4\cdot 21)$$

Suppose, now, the skin density is represented by the single harmonic $\sigma_n S_n$. We see that for large values of n (i.e. for local features), the anomalies Δg and Δg_1 are practically identical, which means that the indirect effect is of no consequence. For small values of n, however, (i.e. for wide-spread inequalities), the indirect effect is material, as can be seen from the following table:

$n =$	1	2	5	10	50
Δg in gals=	·24	·22	·20	·19	·18
Δg_1 in gals=	·00	·07	·13	·15	·18.

This table has been derived by putting $\sigma_n S_n = \rho h S_n$, and assuming $\rho = 2\cdot 67$ gm./cm.^{3} and $h S_n = 1$ mile.

(*b*) *Compensated skin density.*—The results for this case would depend obviously on the type of compensation postulated. Suppose in the first instance, that the skin density $\sigma = \sigma_n S_n$ is compensated according to Pratt's hypothesis, the depth of compensation being τ. The compensation mass is distributed between the spherical surfaces a and $a-\tau$, and its density is given by the usual equation

$$\rho_c = \frac{3a^2}{a^3-(a-\tau)^3} \sigma_n S_n. \qquad \ldots \quad (4\cdot 22)$$

The condition utilized for obtaining this expression is that the mass of the topography and its compensation in any unit column are the same.

The expression for the potential of the coating has been given already; the corresponding potential due to this compensation is

$$V_c = \frac{4\pi f}{2n+1} \cdot \frac{3}{n+3} \cdot \frac{a^{n+3} - (a-\tau)^{n+3}}{a^3 - (a-\tau)^3} \cdot \frac{\sigma_n S_n}{a^{n-1}} \quad \ldots \quad (4\cdot23)$$

Finally, the anomaly $g-\gamma_0$ produced by this topography and its compensation is

$$\Delta g_2 = 4\pi f \cdot \frac{n-1}{2n+1} \sigma_n S_n \left[1 - \frac{3}{a^n(n+3)} \cdot \frac{a^{n+3} - (a-\tau)^{n+3}}{a^3 - (a-\tau)^3} \right]$$

$$= 4\pi f \cdot \frac{n-1}{2n+1} \sigma_n S_n \cdot \frac{n\tau}{2a} \left[1 - \frac{n\tau}{3a} + \frac{n^2 - n - 2}{12} \left(\frac{\tau}{a} \right)^2 \right] \ldots (4\cdot24)$$

We next proceed to the case when the skin density is compensated according to Airy's hypothesis; i.e. the compensation has the same mass as topography in a unit vertical column as before, but is spread over a sphere of radius $(a - \frac{1}{2}\tau)$. The density of compensation is

$$\sigma_c = \sigma_n' S_n, \text{ where } \sigma_n' = \sigma_n \frac{a^2}{(a - \frac{1}{2}\tau)^2} \quad \ldots \quad (4\cdot25)$$

Proceeding as before, we have

$$\Delta g_3 = 4\pi f \cdot \frac{n-1}{2n+1} \sigma_n S_n \left[1 - \left(1 - \frac{\tau}{2a} \right)^n \right] \quad \ldots \quad (4\cdot26)$$

$$= 4\pi f \cdot \frac{n-1}{2n+1} \sigma_n S_n \frac{n\tau}{2a} \left[1 - \frac{n-1}{2} \cdot \frac{\tau}{2a} + \frac{(n-1)(n-2)}{6} \left(\frac{\tau}{2a} \right)^2 + \ldots \right] \quad \ldots \quad (4\cdot27)$$

We will now give another solution based on the condition that the topography and its compensation make the compensation surface $r = (a - \frac{1}{2}\tau)$ an equipotential. The internal potential of coating $\sigma_n S_n$ is

$$\frac{4\pi f}{2n+1} \cdot \sigma_n S_n \frac{r^n}{a^{n-1}},$$

and that of coating $\sigma_n' S_n$ on sphere $r = a - \frac{1}{2}\tau$ is

$$\frac{4\pi f}{2n+1} \sigma_n' S_n \frac{(a - \frac{1}{2}\tau)^{n+2}}{r^{n+1}}.$$

The condition of equality of these for $r = a - \frac{1}{2}\tau$ is

$$\frac{\sigma_n'}{\sigma_n} = \left(1 - \frac{\tau}{2a} \right)^{n-1}. \quad \ldots \quad \ldots \quad (4\cdot28)$$

We see that the law of compensation is quite different from that obtained in $(4\cdot25)$; the corresponding anomaly $g-\gamma_0$ is

$$\Delta g_4 = 4\pi f \frac{n-1}{2n+1} S_n \left[\frac{a^{n+2}\sigma_n - (a - \frac{1}{2}\tau)^{n+2}\sigma_n'}{a^{n+2}} \right]$$

$$= 4\pi f \frac{n-1}{2n+1} \left[1 - \left(1 - \frac{\tau}{2a} \right)^{2n+1} \right] \sigma_n S_n$$

$$= 4\pi f (n-1) \frac{\tau}{2a} \left[1 - \frac{n\tau}{2a} + \frac{n(2n-1)}{3} \left(\frac{\tau}{2a} \right)^2 + \ldots \right] \sigma_n S_n.$$

$$(4\cdot29)$$

The above results may be summarized as follows:

$\Delta g = 4\pi f.\dfrac{n+1}{2n+1}.\sigma_n S_n =$ Direct effect of an uncompensated coating $\sigma = \Sigma\,\sigma_n S_n$ on a sphere of radius $r = a$.

$\Delta g_1 = 4\pi f.\dfrac{n-1}{2n+1}\,\sigma_n S_n =$ Conventional anomaly due to the above coating.

$\Delta g_2 = 4\pi f.\dfrac{n-1}{2n+1}.\sigma_n S_n.\dfrac{\tau}{a}.\dfrac{n}{2}\left(1-\dfrac{n\tau}{3a}\right) =$ Conventional anomaly due to coating $\sigma = \Sigma\sigma_n S_n$, compensated according to Pratt's hypothesis, the depth of compensation being τ.

$\Delta g_3 = 4\pi f.\dfrac{n-1}{2n+1}\sigma_n S_n.\dfrac{\tau}{a}.\dfrac{n}{2}\left(1-\dfrac{\overline{n-1}\cdot\tau}{4a}\right) =$ Conventional anomaly due to coating $\sigma = \Sigma\sigma_n S_n$ compensated according to Airy's hypothesis, the depth of compensation being $\tau/2$.

$\Delta g_4 = 4\pi f.\dfrac{n-1}{2n+1}\sigma_n S_n.\dfrac{\tau}{a}.\dfrac{2n+1}{2}\left[1-\dfrac{n\tau}{2a}\right] =$ Conventional anomaly due to the above coating, compensated in such a way as to make the compensation surface $r = a - \frac{1}{2}\tau$ an equipotential.

The following table gives the maximum values of the anomalies as computed by the above formulæ, due to a coating equivalent to a thickness of 1 mile of rock of normal density 2·67. The values of the constants have been taken as $\sigma_n = \rho h S_n$, $hS_n = 1$ mile $= 160934\cdot26$ cm., $\rho = 2\cdot67$ gm./cm.^{3}, $\tau = 114$ km., $a = 6370$ km. and $f = 6\cdot7\times10^{-8}$.

n	Δg	Δg_1	Δg_2	Δg_3	Δg_4
	gals	gals	gals	gals	gals
1	·2412	·0000	·0000	·0000	·0000
2	·2171	·0724	·0013	·0013	·0032
5	·1973	·1316	·0057	·0058	·0124
10	·1895	·1551	·0130	·0133	·0265
50	·1827	·1755	·0551	·0613	·0876
100	·1818	·1782	·0643	·0888	·0337

From this table we can get an idea of the anomaly expected on a perfectly compensated earth when the topography can be represented by a single harmonic of given amplitude. The figures represent the resultant effects of the attraction of the masses, and the distortion of the level surface. So far as the direct effect of the actual topography and its Hayford compensation is concerned, its magnitude is generally of the order of 30 mgals. For Himālayan stations, however, it can be considerable; in some cases it amounts to 160 mgals, the free-air correction being of the order of 1000 mgals. For stations near the sea also, the correction for topography and its compensation is large, but the height correction in this case is negligible.

(*c*) *Undulations* at the interfaces of different layers of the earth's crust at known depths.*—

The effect of crustal warpings or bucklings at the interfaces can be dealt with in two ways:

(1) By the method of spherical harmonics.

(2) By the formula for attraction of a prism of given cross-section or of a parallelopiped.

We will consider (1) here; formula for (2) will be given later.

Suppose the gravity anomalies $\triangle g = \Sigma\, g_n S_n$ are due to warpings which may be considered as a coating of surface density $\sigma = \Sigma\, \sigma_n S_n$ on a sphere of radius $(a-z)$. The potential due to this coating at a distance r is

$$\delta U = \Sigma \frac{4\pi f}{2n+1} \cdot \frac{(a-z)^{n+2}}{r^{n+1}} \sigma_n S_n ,$$

and the gravity anomaly on sphere of radius a is

$$(\Delta g)_{r=a} = \Sigma\, 4\pi f \cdot \frac{n+1}{2n+1} \left(\frac{a-z}{a}\right)^{n+2} \sigma_n S_n$$

$$= \Sigma\, g_n S_n .$$

Hence $$\sigma_n = \frac{1}{4\pi f} \cdot \frac{2n+1}{n+1} \left(\frac{a}{a-z}\right)^{n+2} g_n. \qquad \dots \quad (4\cdot30)$$

If the volume density at the interface be $\Delta\rho$, we have $\sigma_n = H_n \Delta\rho$, where H_n denotes the amplitude. Knowing $\Delta\rho$ and z, we can find from (4·30) the g_n's corresponding to given H_n's, and vice versa.

As a particular case of the above suppose a surface density $\sigma = \sigma_0 \sin^2 2\theta$ is superposed on a level sphere of radius a. In terms of spherical harmonic functions

$$\sigma = \sigma_0 \sin^2 2\theta = 4\sigma_0 \left(\frac{2}{15} P_0 + \frac{2}{21} P_2 - \frac{8}{35} P_4\right)$$

$$= \Sigma\, A_n P_n .$$

It gives rise to the potential

$$V = 4\pi f \Sigma \frac{1}{2n+1} \cdot \frac{a^{n+2}}{r^{n+1}} A_n P_n$$

$$= 4\pi f a \Sigma \frac{A_n P_n}{2n+1} \quad \text{for } r = a.$$

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* Stoneley, M. N. R. A. S. Geoph. Suppl. **3**, 1933, 176.

Hence $$N=\frac{V}{G}=\frac{4\pi f a}{G}\Sigma\frac{A_n P_n}{2n+1}$$
$$=\frac{3}{\rho_m}\Sigma\frac{A_n P_n}{2n+1}$$
$$=\frac{12\sigma_0}{\rho_m}\left[\frac{2}{15}P_0+\frac{2}{105}P_2-\frac{8}{315}P_4\right]$$
$$=\frac{2\sigma_0}{\rho_m}\left(\frac{24}{35}+\frac{26}{35}\sin^2\theta-\frac{2}{3}\sin^4\theta\right)$$

The corresponding anomaly $(g-\gamma_0)$ is
$$\Delta g=\Delta_1 g-\frac{2Ng}{a}=-\left(\frac{\delta V}{\delta r}\right)_{r=a}-\frac{2Ng}{a}.$$

The case where there are warpings at two interfaces at different depths cannot be solved uniquely unless some assumption is made about the ratio of the amplitude of the two warpings.

(v) *A spherical mass M at depth d.* The direct effect at a point distant $r=\sqrt{x^2+d^2}$ from the centre of the sphere is
$$\Delta_1 g=\frac{fM.\,d}{(x^2+d^2)^{\frac{3}{2}}}.$$

The attraction decreases with the depth according to the inverse square law. The deformation of the level surface due to this mass at a point at height d above the centre of the sphere is
$$N=\frac{V}{G}=\frac{fM}{Gd}.$$

The direct and indirect effects at this point are
$$\Delta_1 g=\frac{fM}{d^2},\ \Delta_2 g=\frac{2GN}{R}=\frac{2fM}{Rd},$$
where R is the radius of the sphere.

Their ratio $\frac{\Delta_1 g}{\Delta_2 g}=\frac{R}{2d}$, a large quantity if the radius of the sphere is taken to be much greater than its depth.

Since we always look for irregularities in the upper layers of the earth's crust (i.e. when d is small), it is always permissible to neglect $\Delta_2 g$. For a given d, the variation of $\Delta_1 g$ with the horizontal distance x is given by the following table :—

x	0	$\pm d$	$\pm 2d$	$\pm 5d$	$\pm 10d$
$\frac{\Delta_1 g}{fM}$	$\frac{1}{d^2}$	$\frac{\cdot 35}{d^2}$	$\frac{\cdot 09}{d^2}$	$\frac{\cdot 008}{d^2}$	$\frac{\cdot 001}{d^2}$

(vi) *A two-dimensional feature.*—Consider a rectangular cylinder with cross-section $x\ y$ as shown in Fig. 9. The cylinder is of infinite extent in the direction perpendicular to the plane of the paper.

The vertical component Z_0 of the attraction of the cylinder at the point O is

$$Z_0 = 2f\rho \left[y \log \tan \frac{x}{y} + \frac{1}{2} x \log \left(1 + \frac{y^2}{x^2}\right)\right]$$

$$= \rho \, y \, F, \qquad \ldots \quad (4 \cdot 31)$$

where $$F = 2f \left[\log \tan p + \frac{1}{2} p \log\left(1 + \frac{1}{p^2}\right)\right],$$

and $$p = \frac{x}{y}. \qquad \ldots \quad (4 \cdot 32)$$

The values of F for different values of p have been tabulated by V. Meinesz*. With the help of this table, the attraction of the cylinder for any position of O can be easily deduced. The chief value of this formula lies in the fact that it enables anomalies to be deduced for the case when the normal structure of the earth is assumed to be three-layered, and when the anomalies are due to intrusion of one layer in another.

(*vii*) *A triangular prism*†.—Suppose we want to find the attraction of a prism at the point C (Fig. 10). Let CAB be the cross-section of the prism through C, and let L_1, L_2 be the lengths of the prism on either side of C. The expression for the potential at C is

$$V = f \rho \, b^2 \sin^2 A \left\{ \cot A \left(\log \frac{2L}{b} + \frac{3}{2}\right) + \cot B \left(\log \frac{2L}{a} + \frac{3}{2}\right) - \angle C \right\}, \ldots \quad (4 \cdot 33)$$

where $$L = \sqrt{L_1 \, L_2}.$$

The vertical attraction of the cylinder at C is

$$\Delta_1 g = 2 f \rho \, b \sin A \left\{ C \cos (A - \phi_1) - \left(\log \frac{b}{a}\right) \sin (A - \phi_1) \right\}. \quad (4 \cdot 34)$$

This formula is especially useful when we are trying to find the attraction of a long mountain ridge. It has been used by Thyssen ‡ to examine the difference between a theoretically calculated and a measured gravity anomaly.

(*viii*) *Attraction of a parallelopiped.*—

This has been dealt with in detail by E. A. Ansel §, who has applied the formulæ to several practical cases.

8. Gravity reductions for deducing subterranean anomalies :—The observed value of gravity at a point on the earth can be made comparable with the normal theoretical value γ_0 by applying certain corrections to it based on different hypotheses. It is not proposed here to go into the merits and demerits of the various reductions usually employed. In this para we will offer some justification for Hayford's isostatic reduction method. Hayford's postulate of local compensation is unreal in the light of

* Gravity Expeditions at Sea, **2**, 1923–32, 24.

† Helmert, Höheren Geodäsie, Vol. II, 277.

‡ Beit. Zu Angewandten Geophysik, **7**, 1939, 366–91.

§ Beit. Zu Angewandten Geophysik, **5**, 1936, 263–95; **6**, 1937, 141–167; **7**, 1939, 21–38.

our modern knowledge about the structure of the earth, and geologists are apt to discard without much ado any results based on this theory. Indeed it has been argued, that whatever success this hypothesis has achieved is only due to the accidental cancellation of different factors. To test this, we will compare the usual isostatic anomalies with those based on other hypotheses for the Himālayan stations. We have chosen mountainous stations for this purpose, because the anomalies there display larger variation.

In the following tables, Δg_A, Δg_B, Δg_C correspond to free-air, Bouguer and the usual isostatic compensation anomalies respectively. On the Airy's hypothesis, anomalies are computed on the assumption that the thickness of the crust corresponding to zero elevation of a region is 40 km. S_1, S_2 are based on V. Meinesz's hypothesis of regional compensation, as described in para 1; Δg_{CH} is the Hayford anomaly on the Helmert spheroid and Δg_{CI} on the International spheroid.

Table A gives the anomalies as well as the means with and without regard to sign for stations in the Kashmīr area. Some of the remaining typical mountainous stations are shown in Table B. Table C gives the anomalies at the three stations Darjeeling, Kurseong and Sandakphu, which lie practically on the same meridian. The ranges of the various anomalies are also shown in a tabular form.

TABLE A

Serial No.	Station	Height	Δg_A	Δg_B	Δg_{CH}	Δg_{CI}	Airy	S_1	S_2
		feet	*cm/sec²*	*cm/sec²*	*cm/sec²*	*cm/sec²*	*cm/sec²*	*cm/sec²*	*cm/sec²*
1	Murree ...	6885	+ ·032	− ·184	− ·025	− ·040	− ·027	− ·024	− ·029
2	Domel ...	2239	− ·167	− ·227	− ·048	− ·063	− ·025	− ·013	− ·010
3	Shādipur ...	5193	− ·116	− ·286	− ·030	− ·045	− ·017	+ ·001	+ ·012
4	Gandarbal ...	5200	− ·094	− ·261	+ ·010	− ·005	+ ·023	+ ·043	+ ·056
5	Hayan ...	6084	− ·105	− ·281	+ ·017	+ ·002	+ ·021	+ ·042	+ ·057
6	Sonāmarg ...	9050	− ·013	− ·296	+ ·043	+ ·028	+ ·078	+ ·065	+ ·084
7	Churawan ...	8151	− ·056	− ·306	+ ·032	+ ·017	+ ·038	+ ·057	+ ·076
8	Minmarg ...	9351	− ·033	− ·324	+ ·035	+ ·020	+ ·040	+ ·058	+ ·078
9	Deosai I ...	13311	+ ·146	− ·298	+ ·090	+ ·075	+ ·094	+ ·128	+ ·153
10	Deosai II ...	12805	+ ·094	− ·332	+ ·062	+ ·047	+ ·065	+ ·102	+ ·126
11	Deosai III ...	12391	+ ·111	− ·301	+ ·095	+ ·080	+ ·100	+ ·135	+ ·160
12	Lālpur ...	5633	− ·045	− ·230	+ ·017	+ ·002	+ ·037	+ ·129	+ ·137
13	Srīnagar ...	5198	− ·070	− ·240	+ ·021	+ ·006	+ ·034	+ ·052	+ ·063
14	Pingalan ...	5227	− 073	− ·245	+ ·012	− ·003	+ ·030	+ ·039	+ ·050
15	Yūs Maidān ...	7867	+ ·024	− ·234	+ ·008	− ·007	+ ·006	+ ·014	+ ·021
16	Korag ...	10952	+ ·149	− ·205	+ ·034	+ ·019	+ ·037	+ ·029	+ ·035
17	Tosh Maidān ...	10315	+ ·135	− ·198	+ 050	+ ·035	+ ·050	+ ·055	+ ·061
Mean without regard to sign =			·086	·262	·037	·028	·042	·058	·071
" with " " " =			− ·005	− ·262	+ ·025	+ ·010	+ ·034	+ ·054	+ ·066

on the surface of compensation S. The physical definition of isostasy demands that the compensation in nature must be so arranged that the topography and compensation make the compensation surface S an equipotential. Hayfordian local compensation fails to satisfy this condition, and the usual isostatic reduction so displaces the masses that hydrostatic equilibrium does not prevail on and below S. To preserve the equality of mass of topography and compensation and in order that the physical condition of isostasy may not be violated, the compensation has to be regional rather than local. Jung* has estimated that the results from rigorous isostatic reduction differ materially from those of the usual isostatic reduction, and hence our usual Hayford anomalies are not suitable for judging equilibrium of the earth as a whole. This is no doubt true, but it should be borne in mind that although Jung's true isostatic reduction satisfies the physical condition of the floating crust, there are several sources of error which still remain. Amongst these may be mentioned the ignoring of the stresses in the earth's crust. Our lack of knowledge of these stresses introduces an element of uncertainty in all the reductions. Again, in reducing the observed gravity on the earth to the level of the geoid, the vertical gradient $\delta g/\delta h$ is allowed for by the normal free-air formula. There is little doubt that this gradient depends on the irregular distribution of visible and buried masses, and varies from place to place. There is also the inevitable uncertainty in the assumed density and depth of of compensation, which militate against a reliable quantitative estimate of the mass anomalies being made.

9. Choice of normal gravity for deducing mass anomalies.—In this chapter we have discussed the methods of estimating mass anomalies with the help of gravity anomalies reckoned from an empirical gravity formula. The question of the gravity formula to be chosen for this purpose merits some consideration. As discussed in chapter II, the formula for normal gravity depends on three parameters† G_e, A and B, which are generally deduced by least squares. We have seen how the value of G_e depends vitally on the distribution of observational material available. For instance, the International formula gives $G_e = 978 \cdot 049$; utilizing the gravity data in India alone (available up to 1928), the value of G_e found‡ was $978 \cdot 021$. For deducing the magnitude of the mass anomalies in an area, it is better to use the value of G_e appertaining to that area only, i.e. for India, we should use the Survey of India value for G_e. This is specially necessary in the case when the gravity anomalies in the limited area are of the same sign. For instance, suppose the mean value of Δg on the Hidden Range is given to be $0 \cdot 02$ mgals, and we are asked to deduce the magnitude of the Hidden Range. If this Δg is computed with a wrong value of G_e, our deduced magnitude of the Hidden Range would be wrong. These remarks also hold, when a profile of Δg is given,

* Zeit für Geoph. **14**, 1938, 27.
† The formulæ comprising the longitude term contain four parameters.
‡ Survey of India, Geodetic Report, Vol. V, 55.

and we want to find the masses responsible for it. A different G_e will reduce all Δg's by the same amount and therefore absolute values of the mass anomalies ΔM will be reduced. It is important to note, however, that the range of variation of ΔM will be correctly depicted, no matter what value of G_c is selected. Also, the effect of choosing slightly different values for the constants A and B in the gravity formula is immaterial when we are dealing with a limited region.

10. Summary.—Gravity anomalies at sea have presented peculiar features, whose interpretation is still not complete. On land, gravity research has afforded valuable clues about the tectonic folding of various regions, and the thickness of the earths' crust therein. On the quantitative side, one can deduce from these anomalies the departures from isostatic equilibrium expressed as a thickness of so many feet of surface coating having the same density as the earths' crust.

By trial and error, it is possible to fit the observed gravity profiles by assuming appropriate mass distributions. Formulæ have been given for the effects of some typical attracting masses.

The question of the gravity reductions and the choice of gravity formula for deducing mass anomalies have also been considered.

Chapter V

STOKES' FORMULA AND THE UNDULATIONS OF THE GEOID

1. Boundary problems of potential theory and geodesy.—In order to be able to understand properly the subject of the undulations of the geoid as derived from gravity anomalies, it is necessary to recapitulate a few facts regarding what are known as the boundary problems of potential theory. These may be enunciated as follows :—

(*i*) Given the value of the potential on the boundary of any surface, find its value at all points of space.

(*ii*) Given the value of gravity at all points of the boundary of a region due to the internal attracting masses, find the potential field at all points of space.

Problem (*i*) may be illustrated by the simple case of a sphere with a coating of surface density σ on it.* Let the known potential at a point (θ, L) on this sphere be

$$V(\theta, L) = \sum_{n=0} Y_n(\theta, L)$$

$$= \sum_{n=0}^{\infty} \left\{ A_n P_n(\mu) + \sum_{m=1}^{n} (A_{nm} \cos mL + B_{nm} \sin mL) P_{nm}(\mu) \right\}, \quad \ldots (5\cdot 1)$$

where $\mu = \sin \theta$.

The coefficients A_n, A_{nm} and B_{nm} in this series expansion are known. If V_i, V_e denote the potentials internal and external to the sphere (assumed to be of radius a), we have

$$\left.\begin{aligned} V_i &= \Sigma \left(\frac{r}{a}\right)^n Y_n(\theta, L) \\ V_e &= \Sigma \left(\frac{a}{r}\right)^{n+1} Y_n(\theta, L) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (5\cdot 2)$$

The skin density of the coating is

$$\sigma = \frac{1}{4\pi} \left\{ \left(\frac{\delta V_i}{\delta r}\right) - \left(\frac{\delta V_e}{\delta r}\right) \right\} = \frac{1}{4\pi a} \sum_{n=0} (2n+1) Y_n(\theta, L).$$

The expressions (5 · 2) for the internal and external potentials can be expressed in a form in which spherical harmonics do not occur. From (5 · 1), by known properties of spherical harmonics, we have

$$Y_n(\theta, L) = \frac{2n+1}{4\pi} \iint V(\theta', L') P_n(\sin \zeta) \, d\mu' \, dL',$$

where $\sin \zeta = \sin\theta \sin\theta' + \cos\theta \cos\theta' \cos(L - L')$, and $\mu' = \sin\theta'$.

* Mac Robert, Spherical Harmonics, 163.

Hence $$V_i = \frac{1}{4\pi}\iint V(\theta', L')\left[\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{a}\right)^n P_n(\sin\zeta)\right] d\mu'\, dL'$$

$$= \frac{a(a^2-r^2)}{4\pi}\int_0^{2\pi}\int_{-1}^{+1}\frac{V(\theta', L')}{(r^2-2ar\sin\zeta+a^2)^{\frac{3}{2}}}\,d\mu'\,dL'$$

Similarly, ... (5·3)

$$V_e = \frac{a(r^2-a^2)}{4\pi}\int_0^{2\pi}\int_{-1}^{+1}\frac{V(\theta', L')}{(r^2-2ar\sin\zeta+a^2)^{\frac{3}{2}}}\,d\mu'\,dL'$$

These formulæ will be made use of in paras 4 and 5.

Problem (ii) is also capable of an easy solution for the case of a sphere. Suppose gravity $g=f(\theta, L)=\Sigma Y_n(\theta, L)$ is known on the sphere. The external potential function V_e is such that $\nabla^2 V_e=0$ in the space external to the sphere, and

$$\left(-\frac{\delta V_e}{\delta r}\right)_{r=a} = f(\theta, L).$$

Taking $V_e=\Sigma\frac{u_n}{r^{n+1}}$, we see that it satisfies the first condition. The second condition gives

$$f(\theta, L) = \Sigma\frac{(n+1)\,u_n}{a^{n+2}} = \Sigma\, Y_n(\theta, L), \text{ or } u_n = \frac{a^{n+2}\,Y_n}{n+1}.$$

Hence $$V_e = \Sigma\frac{a^{n+2}\,Y_n}{(n+1)\,r^{n+1}}.$$

The internal potential V_i can only be written down when the internal masses are known.

The above problems are also soluble when the boundary is a nearly spherical surface, and the attracting masses consist of a coating on it. In geodesy we are mainly concerned with the geoid, which is a level surface of certain masses whose location and extent are not precisely known.

In the problems of potential theory the boundary is not a level surface but its form is known, while in geodesy the reverse is the case; the boundary is a level surface but its form is unknown. The fundamental problem of geodesy is to find the form of a nearly spherical level surface from the variation of gravity on it, it being assumed that there is no attracting matter external to the surface.

Conversely, if the form of a level surface of a system of attracting masses which lie entirely within it is known, the external field can be determined. We shall see that this field is determinable without making any hypothesis about the distribution of matter in the interior of the attracting system.

These problems have been a subject of great controversy on account of the fact that the level surface of the earth with which the geodesist is mostly concerned, namely the geoid, does not enclose all the attracting masses.

2. Stokes' equations.—Chapter I, para 6 gives Stokes' solution of the above problems for a nearly spherical surface. It is shown there that on the level surface

$$r = k\,(1 + u_2 + u_3 + \ldots\ldots\ldots) \qquad \ldots \quad (5 \cdot 4)$$

having no masses external to it, we have

$$g = G\left\{1 + \frac{5}{2}m'\left(\sin^2\theta - \frac{1}{3}\right) + u_2 + 2u_3 + \ldots\ldots\right\}, \ldots \quad (5 \cdot 5)$$

where $G = \frac{fY_0}{k^2} - \frac{2}{3}\,\omega^2 k$ and Y_0 denotes the mass of the attracting body. Equations (5·4) and (5·5), known as Stokes' equations, are of fundamental importance in problems connected with the figure of the earth.

From them we see that given gravity on a level surface having no masses external to it, the parameters defining the shape of the level surface are known. Theoretically, the linear dimension k can also be derived from the variation of gravity on it, but its determination is so weak that it is of no practical value. As particular cases we might mention that if the geoid is a triaxial ellipsoid, we can get its axes and mass from values of gravity at four different latitudes. In the case of a spheroid, we can determine the parameters k, ϵ and M from values of gravity at three known latitudes. As mentioned before, the determination of k is very weak.

The equations (5·4) and (5·5) also afford a solution of the converse problem. They indicate that if we know r, then gravity is known all over the level surface except for one constant. This constant can be determined from the mass of the matter inside the surface, or from the value of gravity at a point on or external to it. In particular, a level spheroid and a triaxial ellipsoid are defined by the constants (k, ϵ) and (k, ϵ, η) respectively. The above amounts to saying that these constants are not sufficient to express the variations of gravity on these surfaces; we need yet another constant G' depending on the mass of the body. The relation between G' and M for the case of an ellipsoid with unequal axes is given by the last of the equations (1·73).

3. Precision of Stokes' equations.— From chap. I we see that formula (5·5) is only correct to the first order of small quantities. More precisely, the value of gravity on a surface of the form (5·4) is given by formula (1·43), which differs from (5·5) in the coefficients of the terms P_2 and P_4. In practice, however, we work not with gravity on a level surface, but with gravity anomalies reckoned from a suitably chosen reference surface. Stokes chose as his reference surface the spheroid

$$r_s = k\left(1 - \frac{2}{3}\epsilon P_2\right),$$

on which gravity is

$$\gamma_0 = G\left\{1 + \left(\frac{5}{2}m' - \epsilon\right)\frac{2}{3}\,P_2\right\}.$$

The separation between the geoid $r = k\,(1 + \sum_{n=2} u_n)$ and this surface is

$$N = k\left\{\frac{2}{3}\epsilon P_2 + u_2 + u_3 + \cdots\cdots\right\}$$

and the gravity anomaly is

$$\Delta g = G\left\{\frac{2}{3}\epsilon P_2 + u_2 + 2\,u_3 + \cdots\cdots\right\}.$$

In these equations it is implied that G has the same value for the geoid and its reference spheroid. Incorporating the term $\frac{2}{3}\epsilon P_2$ in u_2, we obtain the two equations

$$\left.\begin{aligned} N &= k\,\Sigma\, u_n \\ \Delta g &= G\,\Sigma\,(n-1)\,u_n \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (5\cdot6)$$

Stokes' reference surface $r = k\,(1 - \frac{2}{3}\epsilon P_2)$ is not an exact spheroid, but differs from it by $(\frac{23}{63}\epsilon^2 P_2 - \frac{12}{35}\epsilon^2 P_4)$, which can amount to 100 feet in latitude 45°.

There is no objection to taking the above as a reference surface, but Stokes' method of deducing equations (5·6) is open to two objections. One is that he uses an expression for the external potential $V_e = \Sigma\,\dfrac{Y_n}{r^{n+1}}$ in a spherical harmonic series, the convergence of which has been doubted in the region near the boundary of the geoid; the second is that gravity is taken as $-\dfrac{\delta V_e}{\delta r}$, where r is the radius vector at the point considered. Actually $-\dfrac{\delta V_e}{\delta r} = g\cos x$ where x is the angle between the normal and the radius vector at the point considered. The error involved in this, for the case of a spheroid, is $\dfrac{g\,x^2}{2} = 0\,(g\,\epsilon^2) = 0\,(10$ mgals$)$, which is considerable.

To assess the accuracy of equations (5·6), it is more convenient to employ the method of their derivation outlined by Pizetti *. Let the geoid be

$$r_g = k\left[1 - \epsilon \sum_{n=2} Y_n - \epsilon_1 \sum_{n=2} Z_n\right] \qquad \ldots \qquad (5\cdot7)$$

and its reference surface

$$r_s = k\left(1 - \epsilon \sum_{n=2} Y_n\right), \qquad \ldots \qquad (5\cdot8)$$

it being assumed that ϵ_1 is of $0\,(\epsilon^2)$.

For an exact spheroid,

$$\epsilon\,\Sigma\, Y_n = \frac{2}{3}\,\epsilon\left(1 + \frac{23}{42}\,\epsilon\right)P_2 - \frac{12}{35}\,\epsilon^2 P_4.$$

By equations (5·7) and (5·8),

$$r_g - r_s = N = -k\,\epsilon_1\,\Sigma\, Z_n.$$

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* Atti della Reale. Acad. della Scienze di Torino, **46**, 1911.

Let the potential on the reference surface be $U=W_0$, and on the geoid, $W=U+S=W_0$, S being the potential due to the coating between the two surfaces. The mass of this coating is obviously zero. We have

$$g=-\left(\frac{\delta W}{\delta n'}\right)_{\text{geoid}}=-\left(\frac{\delta U}{\delta n'}+\frac{\delta S}{\delta n'}\right)_{\text{geoid}},$$

where

$$-\left(\frac{\delta U}{\delta n'}\right)_{\text{geoid}}=-\left(\frac{\delta U}{\delta n}\right)_{\text{spheroid}}-N\left(\frac{\delta^2 U}{\delta n^2}\right)_{\text{spheroid}}=\gamma_0+N\left(\frac{\delta\gamma_0}{\delta n}\right).$$

Hence

$$g-\gamma_0=N\left(\frac{\delta\gamma_0}{\delta n}\right)-\frac{\delta S}{\delta n'}\doteq-\frac{2\,NG}{k}-\frac{\delta S}{\delta n}\doteq-\frac{2\,S}{k}-\frac{\delta S}{\delta r},\qquad (5\cdot9)$$

where dr denotes the differentiation along the radius vector of a sphere of radius k and G is the mean value of gravity.

If we take $S=\Sigma\, k^{n+1}\,Y_n/r^{n+1}$, we see that

$$\Delta g=\Sigma\,(n-1)\,\frac{Y_n}{k}\text{ and }N=\Sigma\,\frac{Y_n}{G}.$$

These are identical with Stokes' equations.

We have shown in chapter IV, para 4, that in equation (5·9) each term is of order $(g\epsilon^2)$, and terms of $0\,(g\epsilon^3)$ have been neglected. Hence although Stokes derived his equations from first order considerations only, still in the form (5·6), they are really correct to $0\,(\epsilon^2)$. If terms of $0\,(g\epsilon^3)$ were included in equation (5·9), Stokes' expressions for Δg and N would not satisfy it.

This can also be seen as follows. Suppose the geoid is

$$r=k\left[1-\frac{2}{3}\epsilon P_2+\sum_2^{\infty}u_n\right].\qquad\ldots\quad(5\cdot10)$$

By (1·43), gravity on it correct to order ϵ^2 is

$$g=G\left[1+aP_2+\beta P_4+\sum_2^{\infty}\left(n-1\right)u_n\right].\qquad\ldots\quad(5\cdot11)$$

This equation is obtained from the extension* of Green's theorem, that the potential of a rotating attracting mass is

$$U=\frac{1}{4\pi}\iint\frac{gd\sigma}{r}+\frac{\omega^2}{2\pi}\iiint\frac{d\tau}{r}+\frac{1}{2}\omega^2r^2\cos^2\theta,\ \ldots\quad(5\cdot12)$$

and by utilizing the condition that this expression has a constant value on the geoid.

If, now, the reference spheroid is taken as

$$r_s=k\left(1-\frac{2}{3}\epsilon P_2\right)\qquad\ldots\quad(5\cdot13)$$

then

$$\gamma_0=G\,(1+aP_2+\beta P_4),\qquad\ldots\quad(5\cdot14)$$

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* Malkin, Gerl. Beit. zur Geoph., **45**, 1935.

where

$$\left.\begin{aligned} a &= \frac{1}{3}\left(5m' - 2\epsilon\right) + \frac{64}{63}\epsilon m' - \frac{4}{21}\epsilon^2 \\ \beta &= -\frac{4}{35}\left(15\epsilon m' + 2\epsilon^2\right) \\ G &= \frac{fM}{k^2}\left(1 - \frac{2}{3}m' + \frac{4}{9}m'^2 - \frac{8}{15}\epsilon^2 + \frac{4}{9}\epsilon m'\right) \end{aligned}\right\} \quad \ldots \; (5\cdot 15)$$

From these relations we get the usual equations for Δg and $(r - r_s)$.

It is interesting to show that the same relations are obtained even if the equation of the reference spheroid be taken correct to $0(\epsilon^2)$. We have

$$r_s = k\left[1 - \frac{2}{3}\epsilon\left(1 + \frac{23}{42}\epsilon\right)P_2 + \frac{12}{35}\epsilon^2 P_4\right] \quad \ldots \quad (5\cdot 16)$$

Putting

$$u_2 = -\frac{23}{63}\epsilon^2 P_2,\; u_4 = \frac{12}{35}\epsilon^2 P_4 \text{ and } u_3 = u_5 = u_6 = \text{etc.} = 0 \text{ in } (5\cdot 10),$$

we obtain

$$\gamma_0 = G\left[1 + \left(a - \frac{23}{63}\epsilon^2\right)P_2 + \left(\beta + \frac{36}{35}\epsilon^2\right)P_4\right]$$

Hence $\quad r - r_s = k\left[\frac{23}{63}\epsilon^2 P_2 - \frac{12}{35}\epsilon^2 P_4 + \sum_2^\infty u_n\right]$

and $\quad \Delta g = G\left[\frac{23}{63}\epsilon^2 P_2 - \frac{36}{35}\epsilon^2 P_4 + \sum_2 (n-1)\, u_n\right].$

These equations may be written as

$$\left.\begin{aligned} r - r_s &= k\left[\left(u_2 + \frac{23}{63}\epsilon^2 P_2\right) + u_3 + \left(u_4 - \frac{12}{35}\epsilon^2 P_4\right) + \sum_{n=5} u_n\right] \\ \Delta g &= G\left[\left(u_2 + \frac{23}{63}\epsilon^2 P_2\right) + 2u_3 + 3\left(u_4 - \frac{12}{35}\epsilon^2 P_4\right) + \sum_{n=5} u_n\right] \end{aligned}\right\} (5\cdot 17)$$

and are of the same form as $(5\cdot 6)$. We shall see in the next chapter that this is only a particular case of a general theorem on the choice of a reference surface.

It is important to realize that the coefficients of the terms in P_2 and P_4 in formula $(5\cdot 5)$ are not correct to $0(\epsilon^2)$. But the corresponding value of γ_0 is also taken with the same errors in these terms, with the result that these errors cancel out in $(g - \gamma_0)$ and make the equations $(5\cdot 6)$ correct to second order terms. The necessary conditions which the reference surface has to satisfy for the above to be valid will be discussed in the next chapter.

4. Form of the natural geoid, and deviation of the vertical.—A question of prime importance in geodesy is to find the form of the natural geoid of the earth or in other words its deviation N from a reference surface. This can be done by utilizing deflection data or gravity anomalies. The method of determining N from plumb-line deflections is, however, applicable only to a limited area and is used to give the separation of the geoid from a reference spheroid fitting that area best. The gravity anomalies enable N as well as the deflections (η, ξ) to be determined with reference to an absolute spheroid called the 'Earth spheroid', (*cf.* chap. VI). This universal spheroid can also be obtained from deflections, if these were known over the whole globe. But this involves the connection of all the geodetic triangulations of the various countries, which is a remote possibility at present on account of the apparent impossibility of observing deflections on the oceans. The advent of new gravimeters has now made the programme of covering the whole globe with a reasonable mesh of gravity stations well within the range of possibility. We will review here the method of determining N and (η, ξ) from gravity anomalies. Two methods are available :—

(*a*) By a suitable hypothesis, all the masses external to the geoid may be removed. The level surface of the new mass system may be called the corrected geoid and its undulations may be determined from the Δg's. The distance between the natural and corrected geoids is easily calculable from the known mass transfers and when added to the above undulations will give the desired result.

(*b*) The actual topography may be left undisturbed, and the undulations N of the natural geoid may be derived by computing the gravity anomalies on it. We will discuss this method in para 10.

Method (*a*) is the one generally used. The idealisation of the earth which it involves may be performed in several ways as we shall see in para 8. To illustrate the method we will assume that the observed gravity values are reduced isostatically i.e. the effects of topography and its Hayford compensation are removed. The level surface of the new masses is the compensated geoid, and we want to determine its form. Our problem reduces to finding the form of a level surface having no masses external to it, and this is given by the equations (5·6), viz.

for $$\Delta g = G \sum_{2}^{\infty} v_n \qquad \dots \quad (5\cdot 18)$$

we have $$N = k \sum_{2}^{\infty} v_n/n-1. \qquad \dots \quad (5\cdot 19)$$

To get N, we must know the series for Δg in spherical harmonics. Δg is an observational quantity, and some attempts have been made to expand observed gravity anomalies on the globe in a series of spherical harmonics. But it is a very laborious process, and a very large number of terms are required for its adequate representation.

Stokes, however, connected N with Δg by a quadrature formula which is of great practical value. From (5·18) we see that

$$v_n = \frac{2n+1}{4\pi} \iint P_n \frac{\Delta g}{G}\, d\omega,$$

where $d\omega$ represents an element of solid angle on a unit sphere. Substituting in (5·19), we have

$$N = \frac{1}{4\pi} \frac{k}{G} \sum_{n=2} \frac{2n+1}{n-1} \iint P_n \, \Delta g \, d\omega.$$

Now $\Sigma \dfrac{2n+1}{n-1} P_n = 2 \Sigma P_n + 3 \Sigma \dfrac{P_n}{n-1}$

$$= 2\left[\frac{1}{2}\operatorname{cosec}\frac{\psi}{2} - 1 - \cos\psi\right] + 3\left[1 - \cos\psi - 2\sin\frac{\psi}{2} - \cos\psi \log_e\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right].$$

$= 2 f(\psi)$, (say),

and hence
$$N = \frac{k}{2\pi G}\iint f(\psi)\, \Delta g \, d\omega,$$

$$= \frac{k}{2\pi G}\int_0^{2\pi} dA \int_0^{\pi} \Delta g\, f(\psi) \sin\psi \, d\psi, \quad \dots \quad (5\cdot 20)$$

where ψ and A are respectively the angular distance and azimuth of the point at which N is to be determined from the point at which Δg is taken..

Pizetti proved this formula independently starting from the equation

$$\Delta g = -\frac{\delta S}{\delta r} - \frac{2S}{r}. \quad \dots \quad (5\cdot 21)$$

Let
$$\tau_P = 2S + r\frac{\delta S}{\delta r} = \frac{1}{r}\frac{\delta}{\delta r}(S r^2) \quad \dots \quad (5\cdot 22)$$

denote the value of τ at a point P.
Integrating this, we have

$$S r^2 = \int r \, \tau_P \, dr + C.$$

Also by equation (5·3),

$$\tau_P = \frac{a(r^2 - a^2)}{4\pi}\iint \frac{\tau_{sphere}}{d^3} d\omega, \quad \dots \quad (5\cdot 23)$$

where $d^2 = a^2 + r^2 - 2ar\cos\psi$.

Hence
$$S r^2 = \frac{a}{4\pi}\int \tau_{sphere}\int \frac{r^3 - ra^2}{d^3} dr \, d\omega + C.$$

Putting
$$\tau_{sphere} = 2S + a\frac{\delta S}{\delta a} = -a\,\Delta g \quad \dots \quad (5\cdot 24)$$

and integrating with respect to r, we get for $r = a$,

$$S = \frac{a}{2\pi}\iint \Delta g \, f(\psi)\, d\omega,$$

the usual Stokes' quadrature formula.

Formula (5·20) for N enables us also to find deviations of the vertical (η, ξ) at any point O. In Fig. 11, let the gravity anomaly at P be Δg. Then at O we have

$$N = \frac{k}{2\pi G}\iint \Delta g f(\psi)\, d\omega \quad \dots \quad (5\cdot 25)$$

If A is the azimuth of P from O, reckoned positive from south by west, and if η, ξ are the meridional and prime vertical deflections at O, reckoned positive towards south and west respectively, then differentiating (5·25) and simplifying by spherical trigonometry, we get

$$\left.\begin{aligned}\eta &= -\frac{1}{2\pi G}\iint \Delta g \frac{\delta f}{\delta \psi}\cos A \, d\omega \\ \xi &= -\frac{1}{2\pi G}\iint \Delta g \frac{\delta f}{\delta \psi}\sin A \, d\omega\end{aligned}\right\} \quad \ldots \quad (5 \cdot 26)$$

5. Converse problem.—From the preceding formulæ, we can obtain a solution of the converse problem, namely, to find the gravity anomalies from known geoidal undulations. This can be done either by application of the equation (5·9) or from Stokes' equations (5·6).

As in (5·2) the potential $S_e(\theta, L, r)$ at an external point of sphere is

$$S_e(\theta, L, r) = \frac{1}{4\pi}\iint S(\theta', L')\left\{(2n+1)\left(\frac{a}{r}\right)^{n+1} P_n(\cos\zeta)\right\} d\mu' \, dL'$$

$$= \frac{a(r^2-a^2)}{4\pi}\int_0^{2\pi}\int_{-1}^{+1} \frac{S(\theta', L')}{(r^2 - 2ar\cos\zeta + a^2)^{\frac{3}{2}}} d\mu' \, dL', \quad (5 \cdot 27)$$

where $S(\theta', L')$ denotes the potential at a point (θ', L') on the sphere, and ζ is the angle between the directions (θ, L) and (θ', L').

Hence $$\left(\frac{\delta S_e}{\delta r}\right)_{r=a} = \frac{1}{4\pi a}\iint S(\theta', L') \, . \, \frac{1}{4}\operatorname{cosec}^3\frac{\zeta}{2} \, d\mu' \, dL' \quad \ldots \quad (5 \cdot 28)$$

and $$\left(\frac{2S_e}{r}\right)_{r=a} = \frac{1}{4\pi a}\iint \left\{\Sigma(2n+1)P_n\right\} S(\theta', L') \, d\mu' \, dL'$$

$$= -\frac{1}{4\pi a}\iint S(\theta', L') \, d\mu' \, dL'. \quad \ldots \quad (5 \cdot 29)$$

Substituting in (5·9), we have

$$\Delta g = \frac{G}{4\pi a}\iint N\left(1 - \frac{1}{4}\operatorname{cosec}^3\frac{\zeta}{2}\right) d\mu' \, dL'. \ldots \quad (5 \cdot 30)$$

Alternatively,* starting from equations (5·6) and putting $N = k\,F(\theta, L)$, we have

$$u_n = \frac{2n+1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} F(\theta', L') \, P_n(\cos\zeta) \, d\mu' \, dL'.$$

Hence

$$\Delta g = \frac{G}{4\pi}\int_0^{\pi}\int_0^{2\pi} F(\theta', L') \, [1 \cdot 5 P_2 + 2 \cdot 7 P_3 + 3 \cdot 9 P_4 + \ldots] \, d\mu' \, dL'$$

$$= \frac{G}{k \, . \, 4\pi}\int_{-1}^{+1}\int_0^{2\pi} N\left(1 - \frac{1}{4}\operatorname{cosec}^3\frac{\zeta}{2}\right) d\mu' \, dL'. \ldots \quad (5 \cdot 31)$$

It is important to realize that the above formulæ are true only when the geoid and spheroid satisfy the condition of equality of potential

* G.P. Rao, Journal of the Indian Mathematical Society, Vol. 20.

and coincidence of centre of gravity. The undulations of the geoid deduced from astronomic-geodetic deflections cannot be used in (5·31) to give $\triangle g$.

6. Practical application of the above formulæ.— For computational purposes, Stokes' formula (5·20) may be written as follows:—

$$N = \frac{k}{2\pi G}\iint \triangle g f(\psi) \sin\psi \, d\psi \, dA$$

$$= \frac{k}{2\pi G}\int_0^{\pi} F(\psi)\, d\psi \int_0^{2\pi} \triangle g \, dA$$

$$= \frac{k}{G}\int_0^{\pi} \triangle g_0 \, F(\psi)\, d\psi, \qquad \ldots \quad (5\cdot 32)$$

where $\triangle g_0 = \dfrac{\int_0^{2\pi} \triangle g \, dA}{2\pi}$ is the mean value of $\triangle g$ on a circle of radius ψ, and $F(\psi) = f(\psi)\sin\psi$.

If we take zones of width $\triangle\psi_0$, we have

$$N = \frac{k}{G}\triangle\psi_0 \, \Sigma \, \triangle g_m \, F_1(\psi), \qquad \ldots \quad (5\cdot 33)$$

where $\triangle g_m$ is the mean gravity anomaly in a zone, and

$F_1(\psi) = \dfrac{\int_{\psi_1}^{\psi_2} F(\psi)\, d\psi}{\triangle\psi_0} = \dfrac{\Big[\phi(\psi)\Big]_{\psi_1}^{\psi_2}}{\triangle\psi_0}$, ψ_1, ψ_2 being the limits of the zone. Equation (5·33) may now be written as

$$N = \frac{k}{G}\, \Sigma \, \triangle g_m \Big[\phi(\psi)\Big]_{\psi_1}^{\psi_2}. \qquad \ldots \quad (5\cdot 34)$$

The expressions for the various functions are as follows:—

$$\left.\begin{aligned}
f(\psi) &= \frac{1}{2}\Big\{\operatorname{cosec}\frac{\psi}{2} + 1 - 6\sin\frac{\psi}{2} - 5\cos\psi \\
&\qquad - 3\cos\psi \log_e\Big(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\Big)\Big\} \\
F(\psi) &= \cos\frac{\psi}{2} - \frac{1}{2}\sin\psi\Big\{6\sin\frac{\psi}{2} - 1 + \cos\psi\Big[5 + \\
&\qquad 3\log_e\Big(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\Big)\Big]\Big\} \\
\phi(\psi) &= \frac{1}{2}\Big\{1 + 4\sin\frac{\psi}{2} - \cos\psi - 6\sin^3\frac{\psi}{2} - \frac{7}{4}\sin^2\psi \\
&\qquad - \frac{3}{2}\sin^2\psi\log_e\Big(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\Big)\Big\}
\end{aligned}\right\} \quad (5\cdot 35)$$

These functions have been tabulated* by Lambert at intervals of every degree from 0—180°. The following useful table gives the rise of the geoid due to $\triangle g = \cdot 001$ cm./sec.^{2} in different zones.

* U. S. Department of Commerce. Coast and Geodetic Survey. Special Publication No. 199, 114–117.

Elevation of Geoid due to a gravity anomaly of one milligal in each zone.

ZONE	N in ft.	ZONE	N in ft.
$0^\circ - 2^\circ$	+ 1	$70^\circ - 80^\circ$	− 4
2 − 4	+ 1	80 − 90	− 4
4 − 6	+ 1	90 − 100	− 3
6 − 8	+ 1	100 − 110	− 1
8 − 10	+ 1	110 − 120	0
10 − 20	+ 4	120 − 130	+ 1
20 − 30	+ 3	130 − 140	+ 2
30 − 40	+ 1	140 − 150	+ 2
40 − 50	− 1	150 − 160	+ 2
50 − 60	− 3	160 − 170	+ 1
60 − 70	− 4	170 − 180	0

The figures in this table are correct to the nearest foot.

Turning now to deflections, we can write the equations (5·26) in the following form by dividing the space round the origin by a series of concentric zones of width $\Delta\psi_0$.

$$\eta = -\frac{\Delta\psi_0}{G} \operatorname{cosec} 1'' \, \Sigma \left\{ \sin\psi \, \frac{\delta f}{\delta\psi} \times 10^{-3} \right\}_m \left\{ 10^3 \Delta g \cos A \right\}_m \quad (5\cdot36)$$

$$\xi = -\frac{\Delta\psi_0}{G} \operatorname{cosec} 1'' \, \Sigma \left\{ \sin\psi \, \frac{\delta f}{\delta\psi} \times 10^{-3} \right\}_m \left\{ 10^3 \Delta g \sin A \right\}_m . \quad (5\cdot37)$$

The suffix m denotes the mean value of the expression in a zone of width $\Delta\psi_0$, and

$$\sin\psi \, \frac{\delta f}{\delta\psi} = \frac{1}{2} \left\{ -\frac{\cos^2 \psi/2}{\sin \psi/2} - 3 \sin\psi \cos\frac{\psi}{2} + 5 \sin^2\psi \right.$$
$$+ 3 \sin^2\psi \log_e \left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2} \right)$$
$$\left. - 3 \cos\psi \cos^2\frac{\psi}{2} \, \frac{1 + 2\sin\psi/2}{1 + \sin\psi/2} \right\} . \quad (5\cdot38)$$

The values of $\sin\psi \dfrac{\delta f}{\delta\psi}$ for different values of ψ are tabulated below. The table also shows $\left(\sin\psi \dfrac{\delta f}{\delta\psi}\right)_m$ for the various zones.

ψ	$\sin\psi \frac{\delta f}{\delta\psi}$	$\left(\sin\psi \frac{df}{d\psi}\right)_m$	ψ	$\sin\psi \frac{\delta f}{\delta\psi}$	$\left(\sin\psi \frac{df}{d\psi}\right)_m$	ψ	$\sin\psi \frac{\delta f}{\delta\psi}$	$\left(\sin\psi \frac{df}{d\psi}\right)_m$
0°	$-\infty$		8°	−8·9423		70°	−0·1375	
		$-\infty$			−8·2547			+0·2828
$\frac{1}{8}$	−460·2206		10	−7·5671		80	+0·7031	
		−402·7554			−6·6653			+1·0356
$\frac{1}{6}$	−345·2902		15	−5·7635		90	+1·3681	
		−287·5553			−5·3086			+1·5838
$\frac{1}{4}$	−229·8204		20	−4·8537		100	+1·7994	
		−201·6093			−4·5560			+1·8475
$\frac{1}{3}$	−173·3981		25	−4·2582		120	+1·8956	
		−144·7974			−4·0220			+1·5541
$\frac{1}{2}$	−116·1966		30	−3·7837		140	+1·2125	
		− 87·5030			−3·3518			+0·7909
1	− 58·8094		40	−2·9198		160	+0·3692	
		− 44·5176			−2·4705			+0·1846
2	− 30·2257		50	−2·0212		180	0·0000	
		− 21·6836			−1·5475			
5	− 13·1415		60	−1·0737				
		− 11·0419			−0·6056			
8	− 8·9423		70	−0·1375				

It is to be noted that the function $\sin\psi \dfrac{\delta f}{\delta\psi}$ becomes infinite at the origin. This, however, does not make the final result infinite, since $(\Delta g \cos A)_m$ and $(\Delta g \sin A)_m$ in the innermost zone approach $\dfrac{1}{\infty}$ more rapidly than $\left(\sin\psi \dfrac{\delta f}{\delta\psi}\right)_m$ approaches ∞. The following procedure may be employed for the innermost zone, for which ψ is small and $\left(\sin\psi \dfrac{\delta f}{\delta\psi}\right)_m = -\dfrac{1}{2} \operatorname{cosec} \dfrac{\psi}{2}$.

If s denotes the radius of this zone in linear measure, we have $s = k\,\psi$ and

$$\eta = \frac{1}{2\pi G} \iint_0^s \left(\frac{\Delta g \cos A}{s}\right) ds\, dA$$

$$= \frac{s}{2\pi G} \Sigma \left(\frac{\Delta g \cos A}{s}\right) \Delta A.$$

The expression inside the brackets may be evaluated by taking $\Delta A = \dfrac{\pi}{12}$ and estimating $\Sigma \dfrac{\Delta g \cos A}{s}$ in each of the compartments.

From the above table for $\sin\psi \frac{\delta f}{\delta\psi}$, the following table has been deduced, which can be used for getting the deflections at a point from the gravity anomalies.

Meridional deflections due to $\Delta g \cos A = 1$ milligal in each zone, or P.V. deflections due to $\Delta g \sin A = 1$ milligal in each zone.

Limiting radii of zones	$\Delta g \cos A$ or $\Delta g \sin A$	Limiting radii of zones	$\Delta g \cos A$ or $\Delta g \sin A$	Limiting radii of zones	$\Delta g \cos A$ or $\Delta g \sin A$	Limiting radii of zones	$\Delta g \cos A$ or $\Delta g \sin A$
$\frac{1}{2}^\circ - 1^\circ$	$+0''\cdot16$	$10^\circ - 15^\circ$	$+0''\cdot12$	$40^\circ - 50^\circ$	$+0''\cdot09$	$90^\circ - 100^\circ$	$-0''\cdot06$
1 – 2	+0·16	15 – 20	+0·10	50 – 60	+0·06	100 – 120	−0·14
2 – 5	+0·24	20 – 25	+0·08	60 – 70	+0·02	120 – 140	−0·11
5 – 8	+0·12	25 – 30	+0·07	70 – 80	−0·01	140 – 160	−0·06
8 – 10	+0·06	30 – 40	+0·12	80 – 90	−0·04	160 – 180	−0·01

As an example of the application of the preceding formulæ, the effects of the gravity anomalies in India and Europe in raising the geoid at Jubbulpore, Lucknow, a point P with coordinates ($\phi = 25^\circ$, $L = 81^\circ$), Amagaon and Bangalore have been computed and are tabulated below.

Station		N as computed from Indian anomalies feet	N as computed from European anomalies feet
Lucknow	$\phi = 26^\circ\ 55'$, $L = 80\ 59$	$-12\cdot6$	$+1\cdot6$
P_1	$\phi = 25\ 00$, $L = 81\ 00$	$-\ 6\cdot0$	$+1\cdot1$
Jubbulpore	$\phi = 23\ 09$, $L = 79\ 59$	$+\ 3\cdot8$	$+0\cdot9$
Amagaon	$\phi = 21\ 22$, $L = 80\ 28$	$-\ 4\cdot0$	$+0\cdot8$
Bangalore	$\phi = 12\ 58$, $L = 77\ 38$	...	$+0\cdot6$

For India the width of the zones was taken as 2° and for Europe 5°. These latter were divided into compartments 5° square. The effect of the European anomalies varies from 0·6 feet to 1·6 feet, according as the station is in Southern or Northern India.

As a further interesting application of the formulæ, they were used for confirming the considerable differential geoidal rise* between a point in Central India and one in Burma as evidenced by

* Survey of India, Geodetic Report, 1934, Chart XXII.

deflections. A point A ($\phi = 17° 50'$, $L = 78° 50'$) was chosen in Central India, Δg's being available up to a radius of 10° around it. In Burma it was only possible to go up to 4° for the most suitable point B with coordinates ($\phi = 20° 10'$, $L = 96° 10'$). Taking zones of width 1° and applying formula (5·34), the geoidal rise at A was found to be $-18 \cdot 3$ metres and at B $-0 \cdot 75$ metres. The differential geoidal rise is therefore 17·6 metres or 58 feet, which agrees almost perfectly with that deduced from deflections. Deflection results have also indicated an extraordinary rise of the geoid * between Mandalay and Victoria Point. This cannot be corroborated by the gravity method, as data in the south of Siam is rather meagre. More data are needed in the Bay of Bengal and the South China Sea.

Turning now to the wider problem of determining the absolute value of N at different points of the globe, it must be mentioned that it will be a very long time before the requisite data will be available. Hirvonen† utilized all the observed gravity data and supplemented it by extrapolation and by theoretical considerations. He used free-air anomalies, and chose the International spheroid as his reference spheroid. These anomalies were only available for 32% of the northern hemisphere and 4% of the southern. For the remaining areas, for want of better assumption he assumed isostasy to be perfect and used isostatic anomalies. Based on such meagre data, the deduced values of N can only be taken as indicative of the orders of magnitude rather than as true quantities. He deduced N at 62 places on the earth, the results showing a range from $+85$ to -115 metres, i.e. a total range of 200 metres. This would be a very important result if it could be established with certainty, since ideas about the numerical magnitude of N have been very divergent and some people still believe that N can be of the order of 1000 metres. As it is, Hirvonen's values are on the average ± 50 metres, and his discussion on errors shows that their uncertainty is also of the order of ± 50 metres. For numerical work, he proceeded in a method slightly different from that mentioned above. He divided the earth once for all into elementary areas $d\sigma$ (squares of $5° \times 5°$ and $10° \times 10°$), and estimated Δg for each of these. He conceived N at a station to be made up of three parts :—

(*i*) N_R due to Δg's within a radius of 10° from the station; this is called the regional part.

(*ii*) N_c due to Δg's between 10° and 39°; this is the continental part.

(*iii*) N_u due to Δg's between 39° and 180°.

The stations at which he computed N were so chosen that for each of them Δg's up to $\psi = 30°$ were known.

Hunter ‡ estimated that with 1700 stations evenly spaced over the earth's surface, combined with 100 stations suitably distributed

* Survey of India, Geodetic Report, Chart VI.
† Hirvonen, Veröff. des Finnischen Geód: Institutes, No. 19, 1934.
‡ Phil. Trans. of the Royal Soc. of London, Series A, No. 743, 1935, 377–431.

locally, the above formulæ would yield N with a probable error of ± 34 feet and tilt with a probable error of $\pm 0 \cdot 35''$.

Finally it is necessary to emphasize that for computing (N, η and ξ) at a place with the help of the above formulæ, we require a knowledge of Δg all over the globe. This is a desideratum at present. A useful feature of the formulæ, however, is that they enable us to find the effects of different gravimetrically surveyed areas of the globe in producing the rise of the geoid and deflections of the vertical at a given place. As more and more gravity data become available their contributions can be added.

It is obvious from the formulæ that if $\triangle g$ is changed by a constant amount, the derived (N, η, ξ) are unaffected. In other words, the choice of G_e in the normal gravity formula is not important for this purpose.

7. Effect of near zones.—We next proceed to answer the question as to how accurately one can deduce N, η, ξ from $\triangle g$'s by a consideration of the nearer zones only. It does not suffice to argue that the nearer zones are all that are important, because the table on p. 90 shows at a glance that the effects of distant zones are by no means negligible. Indeed it is not difficult to construct an example in which a consideration of zones up to 30° will not give even the sign correctly. Suppose $\triangle g = 20\ P_n$ mgals, where P_n is a zonal harmonic of degree n. Table 1 gives the mean values of these $\triangle g$'s in different zones for $n = 1, 2, 3, 4$ and 10. Table 2 gives the contribution of the gravity anomaly in each zone to the final N. For $n = 1$, we know that the geoidal undulation should be nil. The working in Table 2 shows N to be 4 feet in this case. The discrepancy is due to our zones not being small enough for the computation of the final N. If, however, we had considered the effect of zones up to 30°, we would be in error by over 200 feet. A scrutiny of the table shows that for $n = 2$ and 3, near zones are quite inadequate to give an idea of N. It is obvious from *a priori* considerations that the greater the n, the better the approximation yielded by the nearer zones. This is confirmed by a comparison of the results for P_4 and P_{10}. Table 3, column 1 gives N for the distribution $\triangle g = 20\ P_4$ at a point distant 30° from the pole of P_4. Column 2 gives N for $\triangle g = 20\ P_{10}$ at a point distant 10° from the pole. The error made in deducing N from near zones is markedly greater for $n = 4$ than for $n = 10$. Hence, in order that one may be able to deduce N from a consideration of nearer zones alone (say up to 30°), the $\triangle g$'s should not possess wide-spread inequalities. They should only have harmonics of higher order, so that the effects of remote portions tend to cancel out. As an example, if the gravity anomalies contain the longitude term (second harmonic), near zones will not suffice.

TABLE 1

Zones	Δg_m in mgals				
	$n=1$	$n=2$	$n=3$	$n=4$	$n=10$
0°– 10°	+20	+20	+19	+19	+13
10– 20	+19	+18	+16	+13	− 1
20– 30	+18	+14	+10	+ 5	− 4
30– 40	+16	+10	+ 3	− 3	+ 3
40– 50	+14	+ 5	− 3	− 7	+ 2
50– 60	+11	− 1	− 7	7	− 3
60– 70	+ 8	− 4	− 8	− 3	0
70– 80	+ 5	− 8	− 7	+ 3	+ 3
80– 90	+ 2	− 9	− 2	+ 6	− 2
90–100	− 2	− 9	+ 2	+ 6	− 2
100–110	− 5	− 8	+ 7	+ 3	+ 3
110–120	− 8	− 4	+ 8	− 3	0
120–130	−11	− 1	+ 7	− 7	− 3
130–140	−14	+ 5	+ 3	− 7	+ 2
140–150	−16	+10	− 3	− 3	+ 3
150–160	−18	+14	−10	+ 5	− 4
160–170	−19	+18	−16	+13	− 1
170–180	−20	+20	−19	+19	+13

TABLE 2

Zones	N in feet				
	n = 1	n = 2	n = 3	n = 4	n = 10
0°– 10°	+ 87·6	+ 88·2	+ 84·0	+ 84·0	+ 56·7
10– 20	+ 76·9	+ 71·4	+ 63·0	+ 52·5	– 4·2
20– 30	+ 47·5	+ 37·8	+ 27·3	+ 12·6	– 10·5
30– 40	+ 13·7	+ 8·4	+ 2·1	– 2·1	+ 2·1
40– 50	– 14·7	– 6·3	+ 4·2	+ 8·4	– 2·1
50– 60	– 31·3	+ 2·1	+ 18·9	+ 18·9	+ 8·4
60– 70	– 30·9	+ 14·7	+ 29·4	+ 10·5	0·0
70– 80	– 20·6	+ 31·5	+ 27·3	– 12·6	– 12·6
80– 90	– 6·3	+ 33·6	+ 8·4	– 23·1	+ 8·4
90–100	+ 4·8	+′ 27·3	– 6·3	– 16·8	+ 6·3
100–110	+ 8·6	+ 12·6	– 12·6	– 4·2	– 4·2
110–120	+ 3·6	+ 2·1	– 4·2	+ 2·1	0·0
120–130	– 9·7	0·0	+ 6·3	– 6·3	– 2·1
130–140	– 23·7	+ 8·4	+ 4·2	– 12·6	+ 4·2
140–150	– 34·2	+ 21·0	– 6·3	– 6·3	+ 6·3
150–160	– 34·2	+ 27·3	– 18·9	+ 10·5	– 8·4
160–170	– 24·4	+ 23·1	– 21·0	+ 16·8	– 2·1
170–180	– 8·4	+ 8·4	– 8·4	+ 8·4	+ 6·3
TOTALS ...	+ 4·3	+ 411·6	+ 197·4	+ 140·7	+ 52·5

TABLE 3

ZONES	N in feet		ZONES	N in feet	
	n = 4	n = 10		n = 4	n = 10
0°–10°	+ 4·2	+ 16·8	90–100	– 6·3	+ 2·1
10–20	0·0	– 4·2	100–110	– 2·1	– 2·1
20–30	– 2·1	– 2·1	110–120	0·0	0·0
30–40	– 2·1	0·0	120–130	0·0	0·0
40–50	+ 2·1	0·0	130–140	– 4·2	+ 2·1
50–60	+ 2·1	+ 2·1	140–150	– 4·2	+ 2·1
60–70	0·0	0·0	150–160	– 2·1	– 2·1
70–80	– 4·2	– 4·2	160–170	0·0	– 2·1
80–90	– 8·4	+ 4·2	170–180	0·0	+ 2·1

For $n = 4$, total $N = -27\cdot3$ feet
" $n = 10$, " $= +14\cdot7$ "

The same holds for the deflections, as can be seen from the following investigation. Let the geoid be taken as the triaxial ellipsoid

$$r=k\left\{1+\epsilon\left(\frac{1}{3}-\sin^2\theta\right)+C\cos 2\,(L-L_0)\cos^2\theta\right\},$$

and let its reference spheroid be

$$r=k\left\{1+\epsilon\left(\frac{1}{3}-\sin^2\theta\right)\right\}.$$

The gravity anomaly is now represented by the systematic longitude term

$$\Delta g=GC\cos 2\,(L-L_0)\cos^2\theta.$$

Obviously $\quad N=kC\cos 2\,(L-L_0)\cos^2\theta,$

$$\eta=-\frac{\delta N}{k\delta\theta}=C\cos 2\,(L-L_0)\sin 2\theta,$$

$$\xi=-\frac{\delta N}{k\cos\theta\,\delta L}=2\,C\sin 2\,(L-L_0)\cos\theta.$$

Heiskanen gets $C=19\times 10^{-6}$, $L_0=0°$. Using these values, we get for Kaliānpur (latitude 24°, longitude 78°) $\eta=-2''\cdot 6$, $\xi=+2''\cdot 9$. Using the table on p. 92, we see that the effect of these gravity anomalies comprised within a radius of 15° round Kaliānpur is $\eta=-0''\cdot 43$, $\xi=+0''\cdot 45$, so that the outer zones beyond this are responsible for over 2″ in each compartment. This residual error may be much greater in an extreme case. Thus at latitude 24°, longitude 45°, the deflection would be $\eta=0$, $\xi=2\,C\cos 24°=7''$. The assumed gravity anomalies within a radius of 15° produce a prime vertical deflection of about +1″ only, showing that the outer zones are most important.

It should be mentioned, however, that both for vertical separation and for deviations, the relative deflections of two points not too far from each other can be obtained from a consideration of the near zones alone, the effect of the remote anomalies being nearly the same for both.

A point worth mentioning is that even in the absence of systematic error in Δg's, the effect of outer zones is considerable in the case of N, and a fair knowledge of Δg over the whole globe is required before N can be determined with any degree of accuracy. The deviations are however not so sensitive. Thus, supposing the Tibetan plateau to be an area of gravity anomaly $-0\cdot 020$ gals and to be bounded by latitudes 30°–36° and longitudes 80°–100°, it was found by an application of the table on p. 92 that it would produce a meridional deflection of $+0''\cdot 3$ and a prime vertical deflection of $+0''\cdot 4$ at Kaliānpur. These are surprisingly small, considering the large anomaly assumed and its large extent.

An attempt has been made to find out the orientation of the International and Helmert spheroids at Kaliānpur from the gravity anomalies in India. Gravity data was available up to a radius of 10° from Kaliānpur, and the average values of $\Delta g\cos A$ and

$\Delta g \sin A$ were computed in suitably chosen zones. The deflections with respect to Helmert's spheroid came out to be $\eta_H = +1''\cdot 6$, $\xi_H = +4''\cdot 2$, and with respect to the International spheroid as $\eta_I = +1''\cdot 3$, $\xi_I = +4''\cdot 0$. These differ by about $1''$ from the deflections adopted at Kaliānpur H.S. (serial no. 240, Supplement to Geodetic Report, Survey of India, Vol. VI). A much more reliable value of the deflections would be obtained if Δg's were known in the Arabian Sea (between latitudes 12° and 20°), in the Bay of Bengal, and in Burma, Siam and Tibet.

It can be argued that although a consideration of near zones may not give the absolute magnitude of N correctly, it may suffice for giving the relative changes of N for near points, because the effect of remote portions will be much the same. We have seen already that the gravity anomalies in Europe produce a differential effect of 1 ft. for points in Northern and Southern India. This is by no means serious, but Europe is only a small portion of the earth's surface and it is conceivable that the differential effects of the whole globe may be much greater.

A good use of the knowledge of Δg's in near zones is as follows. For obtaining N from deflections, we require such a close mesh of astronomic-geodetic stations that the deflections at intermediate points should be interpolable. To provide sections of the geoid in the plains of India, recent observations have been made at intervals of 10 to 15 miles. In mountainous regions, a much closer interval is needed. Where, however, the deflection stations are few, and cannot be interpolated, it is possible to supplement them by additional deflections obtained with the help of Δg's in near zones. The only condition required is that Δg's should be known within an area of such an extent round the station that the deflections due to the remote portions at points in our limited area are linearly interpolable. We may proceed as follows:—

Divide the earth into two parts A, B by a circle of radius 150 to 200 miles surrounding the station. Suppose astronomic-geodetic deflections (η, ξ) are known at points a_1, a_2, a_3 ...... Compute at these places $(\delta\eta_A, \delta\xi_A)$ due to Δg's in region A. The differences $(\eta - \delta\eta_A, \xi - \delta\xi_A)$ are due to effects of Δg's in region B and to the inclination of the gravity spheroid to the triangulation spheroid. The latter quantity varies very slowly from point to point, and we can assume it to be interpolable. We have also so chosen the boundary of region A that deflections due to Δg's in region B are interpolable. Hence from the known points these differences can be interpolated for all the points at which deflections are required. Adding to these differences the deflections $(\delta\eta_A, \delta\xi_A)$ due to Δg's in region A we have the final deflections. The method thus consists in first removing the local effects as best as possible, and then interpolating and adding on the local effects.

This method can also be used for interpolation of observed deflections even without the aid of gravity data. As a rule, observed deflections, especially in mountainous areas, are not amenable to interpolation. For instance, we have at

Dehra Dūn ... $\eta = -32''\cdot 2$,
Rājpur ... $\eta = -42''\cdot 2$,
Mussoorie ... $\eta = -31''\cdot 2$.

Simple interpolation for Rājpur, which lies practically midway between Dehra Dūn and Mussoorie, would give an error of about 10″. We can, however, predict η at Rājpur to the nearest second by computing the Hayford deflections ($\delta\eta_c$, $\delta\xi_c$) due to topography in the surrounding area, say 100 miles round the station.

We find $\delta\eta_c = -15''\cdot 15$ at Dehra Dūn,
$= -15''\cdot 08$ at Mussoorie,
$= -24''\cdot 3$ at Rājpur,
wherefrom $(\eta - \delta\eta_c) = -17''\cdot 0$ at Dehra Dūn,
and $(\eta - \delta\eta_c) = -16''\cdot 1$ at Mussoorie.

Hence by interpolation, at Rājpur we have

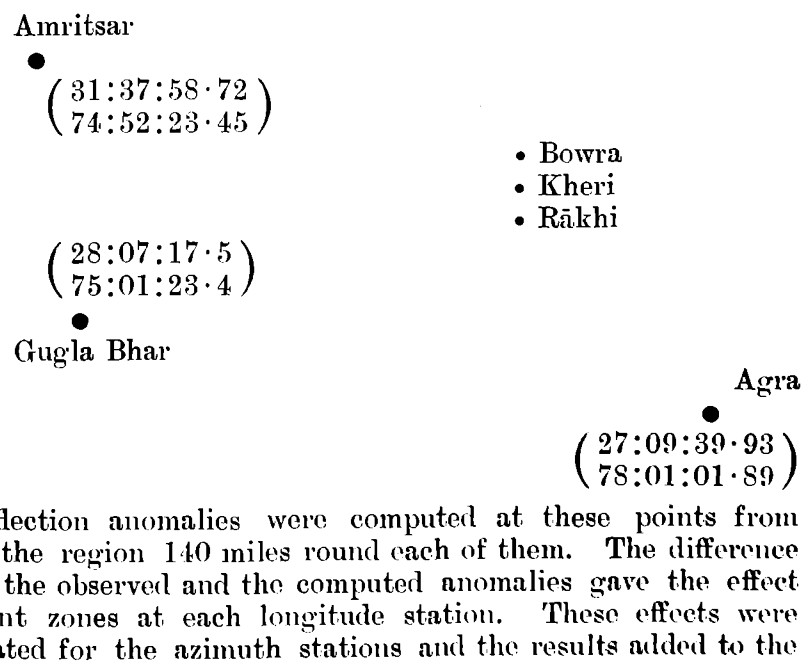
$$(\eta - \delta\eta_c) = -16''\cdot 6$$

or

$$\eta = -24''\cdot 3 - 16''\cdot 6 = -40''\cdot 9,$$

which differs from the observed value by 1″·3 only.

We will now mention how use was made of this method in a practical case*. The problem was to obtain Laplace azimuths at three stations Bowra, Kheri and Rākhi of a triangulation series. Prime vertical deflections ξ were available at the widely separated points Amritsar, Gūglā-Bhar and Agra. Direct interpolation was therefore inadmissible. The relative positions of the stations are roughly shown below.



P.V. deflection anomalies were computed at these points from Δg's in the region 140 miles round each of them. The difference between the observed and the computed anomalies gave the effect of distant zones at each longitude station. These effects were interpolated for the azimuth stations and the results added to the deflection anomalies produced by the near zones.

* Prof. Paper 28 of the Survey of India, p. 56.

8. Undulations of the geoid.—The problem of determining the possible magnitudes of N and N_c, the elevations of the natural and compensated geoids respectively above their reference spheroids, has attracted considerable attention of geodesists, and is still a live subject for research. Two methods by which the above can be determined, namely, from gravity anomalies and plumb-line deflections, have already been mentioned. Both these involve actual observations on the earth. The undulations can also be deduced theoretically by making some assumption about the internal constitution of the earth. As example of this may be mentioned Helmert's* computations for finding the effect of the uncompensated continental masses in producing the warping of the geoid. He came to the conclusion that the undulations could be of the order of $\pm$ 1000 m. Utilizing his expansion of the lithosphere in terms of spherical harmonics, Prey† has also estimated the possible undulations for a non-isostatic earth. His work confirms Helmert's results, his N's ranging from -4000 to $+4000$ feet.

The warps of the geoid have also been estimated for an isostatic earth by Prey† and Jung‡. Taking the first seven terms in Prey's development of the lithosphere, Jung showed that the range of N for an isostatic earth is about 300 feet. Prey's results also exhibit much the same range.

As regards the determination of N from observed gravity, we might mention the work of Helmert §, Ackerl || and Hirvonen. With the meagre data at his disposal, Helmert used free-air anomalies and estimated the maximum value of N to be of the order of $\pm$ 100 m. Ackerl reduced about 4000 gravity stations by Prey's reduction, and using Brun's formula $N = \frac{2a}{3G} \Delta g_p$ obtained undulations of the order of 2800 m. Jung¶ has discussed in detail the fallacy of his reasoning, and says that he gets these large N's due to using an incomplete formula.

Ackerl next expressed the gravity anomalies of the earth, reduced according to Prey's reduction, in spherical harmonic functions up to terms of the 16th order $. From these anomalies he computed N's by utilizing Stokes' equations**. His results are set forth in Table II, p. 265 of his paper, and show undulations of the order of 1000 m. The largest depression comes out to be 837 m. in the Pacific

* Höheren Geodäsie, **2**, 1884, Chap. IV.

† Prey, Gerl. Beit. **36**, 1932, 242–68.

‡ Jung, Zeit. f. Geoph. **8**, 1932, 51.

§ Helmert, Die Schwerkraft und die Massenverteilung der Erde. Encycl. d. math wiss., VI, 1, 7, Abschnitt 10.

|| Ackerl, Gerl Beit. **29**, 1931, 273–335.

¶ Jung, Gerl. Beit. **36**, 1932, 212.

$ F. Ackerl. Das Schwerkraftfield der Erde. Akad. wien sitz.-ber. d. mathem. naturw. kl. (IIa), **140**, 1931 und **141**, 1932.

** F. Ackerl: Die Ergebnisse der Entwicklung des Schwerkraftfeldes der Erde nach kugelfunctionen bis zur 16 ordnung. Zeit. f. geoph. **9**, 1933, 273.

Ocean at $\phi = 11^\circ$, $L = 22^\circ$. Ackerl assessed the accuracy of his deduced N's to be $\pm$ 50 m. and affirmed that the magnitude and distribution of these undulations showed that the geoid cannot be represented sufficiently accurately by a triaxial ellipsoid. Much has been written by Hopfner in justification of these results. It is enough to point out here that Ackerl's results have to be rejected, his work being vitiated by the fact that Prey's anomalies cannot be applied to Stokes' formula as it stands. The necessary modification required will be indicated later in para 10.

An attempt to determine N from the observed gravity anomalies, which rests on correct theory but is handicapped by dearth of observational material, is that of Hirvonen * already mentioned in para 6. On account of lack of data, he had to resort to highly precarious interpolations and extrapolations in estimating the gravity anomalies. His work however brings to light that the elevations are on the average $\pm$ 50 m., and refutes the possibility of undulations of 1000 m.

9. Reductions for finding the form of the natural geoid.—In the proof of equations (5·6), we have postulated that by a suitable reduction all the masses external to the geoid have been removed. One such reduction is the isostatic one, in which the masses protruding above the geoid are abolished. The level surface of the new mass-distribution is the compensated geoid, and equation (5·20) gives the rise N_c of this geoid above its reference spheroid. To get the rise of the natural geoid, we have to add the separation between the two geoids due to the mass transfers. This is easily computed with the help of Lambert and Darling's † tables for determining the form of the geoid.

Jeffreys ‡ has shown in an elegant way, that although there are masses outside the natural geoid, equation (5·20) is valid to the first order in height of the earth above the geoid, if we use values of gravity reduced to the natural geoid by free-air. The reason for the propriety of free-air reduction in Stokes' formula is as follows :—

Imagine all the topography above the natural geoid to be condensed on the geoid. This is called the condensation reduction, and we will designate the level surface of the new mass distribution as the condensed geoid. It can be easily shown that for all practical purposes, so far as N is concerned, the natural and condensed geoids may be considered as identical. Thus by Lambert's tables, the geoidal rise due to a cap of thickness 3 km., radius 100 km. and density = 2·8, is 34·8 metres. If the mass of this cap be considered as a coating, the rise due to it amounts to

$$N = \frac{V}{G} = \frac{3}{2a} \cdot \frac{ch\rho}{\rho_m} \left(1 - \frac{h}{c} + \frac{h^2}{2c^2} \right) = 34 \cdot 2 \text{ metres.}$$

In the above formula $h\rho = 2 \cdot 67 \times 3 =$ surface density of the coating, $c = 100$ km. = horizontal extent and $\rho_m =$ mean density of the earth.

* Veroffent. des Finnischen geodatischen Institutes, No. 19, Helsinki. 1934.

† U.S. Coast and Geodetic Survey, Special Pub. No. 199.

‡ Gerl. Beit. z. Geophysics **36**, 1932, 206–11.

This shows how closely identical the effects of the actual and condensed topography are, even for the unfavourable case that has been considered. This is due to the fact that N depends more on the actual amount of the attracting mass than on its configuration. If, then, we can get Δg's on the condensed geoid, these can be used in Stokes' equation for getting the rise of the natural geoid, since there are no masses external to the condensed geoid.

Now, let E be a point on the earth (Fig. 12), and A the corresponding point on the geoid. Let the masses inside the geoid be designated by M, and the masses between the geoid and the earth's surface by m. Before condensation, g_E = attraction of masses M at E + attraction of masses m at E, while after condensation, g_A = attraction of masses M at A + attraction of condensed masses m at A. The condensation reduction is, therefore, $g_A - g_E = 2gh/k +$ (attraction of condensed masses m at A − attraction of masses m at E)

$$= \frac{2gh}{k} + \left(\iint \frac{\sigma dS}{r} - \iiint \frac{dm}{r} \right),$$

where σ denotes the skin density, and the volume integral extends throughout the mass m. The term in brackets on the right hand side can be evaluated rigorously with the help of Hayford's reduction tables, but for our purpose we may neglect the curvature of the earth and regard the masses m between E and A as an infinite plateau. Then this term vanishes, and we have $g_A = g_E + 2gh/k$, which is nothing else but Jeffreys' result that Δg_A's need only be used. In mountainous areas, however, we may regard the topography above A as an infinite plateau on which are superposed some undulations. After condensation, the effect of the infinite plateau cancels out and we are left with the so-called "Gelande-Reduction" Δg_R. Hence a more correct expression for g_A is

$$g_A = g_E + \Delta g_R + 2gh/k.$$

Δg_R is always positive. Its values for some of the typical mountain stations in India are as follows:

Station		Altitude feet	Δg_R gals
Domel	...	2239	·015
Hayan	...	6084	·028
Sonamarg	...	9050	·021
Churawan	...	8151	·024
Minmarg	...	9351	·023
Wozul Hadur	...	13921	·019

Of course there are some mountain stations for which Δg_R is less, but 0·020 gals seems a fair average value to take for uneven topography. If, then, we neglect Δg_R and obtain N from Δg_A's, we are making a systematic error of about 20 mgals in all the mountainous regions. A casual error of this amount in (say) every degree square will not have much effect on the resulting value of N, but it is not desirable to have such a large systematic error for all the mountainous regions of the globe. These remarks hold only for determining N. If the objective is to determine the ellipticity of the level surface, free-air gravity anomalies can be used without objection.

The above shows that for the determination of N for practical purposes, it is simplest to use condensation reduction. The use of isostatic reduction entails an extra step, namely, the computation of the deformation of the natural geoid due to mass displacements. The relative merits of these two reductions for determining N have been considered by the author* in a paper entitled "Gravity reductions and the figure of the earth". W. D. Lambert† has also contributed several articles on the subject which are interesting.

Another point of view about the determination of N has been put forward by Hopfner ‡ in various articles. He vigorously denounces all other reductions except Prey's, and asserts that this is the only reduction which can be used for determining the geoidal rise. In this reduction the earth is left as it is, and values of g are deduced on the natural geoid as if gravity observations were made there. It is obvious that to make use of Preys' anomalies we have to extend formula (5·20), so as to be applicable to the case when there are masses external to the geoid. The formula can be modified for Bouguer anomalies as well.

10. Extension of Stokes' theorem.—We will now consider method (*b*) mentioned in para 4. In this the mass distribution of the earth is not interfered with, and the problem is to get a value of the potential at a point inside the attracting masses. The appropriate formulæ have been worked out by Malikin § and Lambert. || The integral equation between N and Δg, when there are some masses external to the geoid, is

$$gN - 2U_e = \frac{1}{2\pi}\iint \frac{\Delta g \, d\omega}{r} + \frac{1}{2\pi}\iint \frac{3}{2k}\, g\, N \frac{d\omega}{r}$$

$$= -\frac{6\rho G}{\rho_m}\left(h_0 + \frac{h_1}{3}\right) + \frac{1}{4\pi}\iint (k\Delta g + 3\,U_e)\, F(\psi)\, d\omega,$$

where U_e denotes the potential of the masses between the geoid and the earth. Prey's anomalies can appropriately be used in this formula. We see that as in the case of no external masses, a knowledge of the distribution of density inside the geoid is not required. But to get U_e and Δg it is essential to know the precise arrangement of masses external to the geoid. Hence if there are masses inside and outside a level surface, its form cannot be determined from a mere knowledge of the values of gravity on it. It is essential to know the external masses as well. The situation is therefore precisely the same as when there are no external masses.

* Gulatee, Gerl. Beit. z. Geoph. **53**, 1938, 332-36.
† Lambert, Bull. Geod. No. 41, 1934, 26-33
‡ Gerl. Beit. z. Geoph. **38**, 1933, 309-20.
§ Gerl. Beit. z. Geoph. **45**, 1935, 133-147.
|| Gerl. Beit. z. Geoph. **49**, 1937, 199-209.

The rigid computation of Prey's anomaly is by no means less troublesome or less inaccurate than that of Hayford's anomaly, and there is no particular advantage in using it for the computation of N. It might however be put to the following two uses:—

If S is the natural geoid, and g_P the value of gravity on it (due to actual topography), then $\iint g_P \, dS = 4\pi M$, where M is the sum of the masses inside the geoid.

Again if V_1 denotes the potential due to the internal masses, then

$$V_1 = \frac{1}{4\pi}\iint g_P \frac{dS}{r} + \frac{\omega^2}{2\pi}\iiint \frac{d\tau}{r}.$$

Hence if g_P is known, we can obtain the total masses inside the geoid as well as their potential without knowing the internal law of density. From the point of view of the geoidal rise, this reduction has received exaggerated importance at Hopfner's hands.

11. Summary.—In this chapter, the subject of deriving the undulations of the geoid and plumb-line deflections from the gravity anomalies is considered from both the theoretical and practical aspects. It is explained that Stokes' formula can only be used for certain reductions, and that it has to be suitably modified before it can be applied to Bouguer and Prey's reductions.

Examples have been given to illustrate the gravity method of determining (N, η, ξ) from the available gravity data. Besides these, a general idea is given of the maximum possible separation of the geoid from its reference surface.

Chapter VI

CHOICE OF A REFERENCE SURFACE FOR GRAVITY WORK

1. Reference surface.—The earth's surface is very irregular and cannot be expressed by a simple mathematical formula. The same holds for the natural geoid, and in dealing with problems connected with the figure of the earth, it is therefore customary to define the geoid with respect to some suitably chosen reference surface.

Any surface may be taken as a reference surface, but it is advantageous if it be so chosen as to fit the geoid reasonably well. It may be determined from physical considerations, or may be defined geometrically. In triangulation the latitudes and longitudes are computed on a reference surface which is a true spheroid. We shall see presently that considerations which determine the reference surface for gravity work are quite different from those necessary in the case of triangulation and arc measurements.

2. Nearly spherical surface.—For a proper understanding of the various definitions of the reference surfaces in gravity work it is essential to know the interpretation of the various harmonic terms in the equation of a nearly spherical level surface.

Let the geoid be

$$r=a\ (1+Y_0+Y_1+Y_2+Y_3+\dots\dots) \quad \dots \quad (6\cdot1)$$

where Y_0 is a constant,

$$Y_1 = A_1 \sin\theta + (A_{11}\cos L + B_{11}\sin L)\cos\theta,$$

$$Y_2 = \frac{A_2}{2}(3\sin^2\theta - 1) + (A_{21}\cos L + B_{21}\sin L)\frac{3}{2}\sin 2\theta$$

$$+ (A_{22}\cos 2L + B_{22}\sin 2L)\, 3\cos^2\theta,$$

and so on.

θ, L are the geocentric latitude and longitude respectively of a point on the surface.

The volume of this surface is

$$V=\frac{4\pi a^3}{3}(1+Y_0)^3+\frac{4\pi a^3}{3}(A_1^2+A_{11}^2+B_{11}^2)+\dots\dots$$

$$=\frac{4\pi a^3}{3}(1+Y_0)^3, \text{ if } A_1^2,\ A_2^2 \text{ etc. can be neglected.}$$

Obviously, the radius of a sphere of equal volume is $k=a\ (1+Y_0)$.

As an example, the equation of a spheroid, neglecting terms of order ϵ^2, is

$$r=a\left(1-\frac{1}{3}\epsilon-\frac{2}{3}\epsilon P_2\right). \quad \dots \quad (6\cdot2)$$

We have $k = a\left(1 - \frac{1}{3}\epsilon\right), Y_0 = -\frac{1}{3}\epsilon$ and $A_2 = -\frac{2}{3}\epsilon$.

For a true spheroid, the value of k correct to the ϵ^4 term is given by the expression

$$k = a\left(1 - \frac{1}{3}\epsilon - \frac{1}{5}\epsilon^2 - \frac{13}{105}\epsilon^3 + \frac{205}{504}\epsilon^4\right).$$

We see from the above that the surface $r = k\,(1 + Y_2 + Y_3 + \ldots)$, the spheroid $r = k\,[1 + \epsilon\,(\frac{1}{3} - \sin^2\theta)]$ and the sphere $r = k$ have the same mass to a high degree of approximation if

$$Y_2 = \epsilon\left(\frac{1}{3} - \sin^2\theta\right) + \left(A_{21}\cos L + B_{21}\sin L\right)\frac{3}{2}\sin 2\theta + \ldots$$

and if terms of order ϵ^2, ϵ^3 etc. are neglected.

Again, the centre of gravity of mass or volume of surface (6·1), assuming it to be homogeneous, is given by

$$\bar{z} = \frac{\iint z\,dm}{M}$$

$$= a\,A_1 + \frac{4\pi a^4}{5M}\,A_1\left(A_1^2 + A_{11}^2 + B_{11}^2\right) + 0\left(a\,A_1^5.\right)$$

$$= aA_1, \text{ terms } aA_1^3 \text{ etc. being negligible;}$$

similarly $\bar{x} = aA_{11}$, and $\bar{y} = aB_{11}$.

If, however, the surface be considered as a sphere of radius a and density ρ_m, overlain by a coating $a\rho\,(Y_1 + Y_2 + \ldots)$, we have

$$\bar{x} = a\,A_{11}\cdot\frac{\rho}{\rho_m},\ \bar{y} = a\,B_{11}\cdot\frac{\rho}{\rho_m},\ \bar{z} = a\,A_1\cdot\frac{\rho}{\rho_m}.$$

If the origin be chosen at the centre of gravity of volume of the surface, the equation of the surface becomes

$$r = k\,(1 + Y_2 + Y_3 + \ldots) \qquad \ldots \quad (6\cdot3)$$

This is the reason why the Y_1 term is absent from equation (6·2) which is the equation of a spheroid referred to its centre as origin.

The Y_2 term is very important as it enables the ellipticity of a level surface to be defined. We know that $r = k\,(1 - \frac{2}{3}\epsilon P_2)$ is a spheroid of ellipticity ϵ, and

$$r = k\left[1 - \frac{2}{3}\epsilon P_2 + \frac{1}{2}\eta\cos^2\theta\cos 2\left(L - L_0\right)\right] \qquad \ldots \quad (6\cdot4)$$

is an ellipsoid, the mean ellipticity of whose meridians is ϵ and the ellipticity of whose equator is η. The actual ellipticities of the ellipsoid (6·4) in the planes xz and yz are $\epsilon - \eta/2$ and $\epsilon + \eta/2$. This shows that the last term in equation (6·4) contributes to the meridional ellipticity, but averages out to zero in the mean meridional ellipticity.

Equation (6·3) may be written as

$$r = k \left\{ 1 - \frac{2}{3}\epsilon P_2 + \frac{3}{2}\left(A_{21}\cos L + B_{21}\sin L\right)\sin 2\theta \right.$$
$$\left. + 3\left(A_{22}\cos 2L + B_{22}\sin 2L\right)\cos^2\theta + \sum_{n=3} Y_n \right\}. \quad \dots (6\cdot5)$$

ϵ may be defined as the mean meridional ellipticity of the geoid. It must be noted, however, that this definition can only hold under certain reservations because there are certain other terms which contribute to the meridional ellipticity. As an example, the coefficient A_4 in the fourth harmonic

$$Y_4 = \frac{A_4}{8}\left(34\sin^4\theta - 30\sin^2\theta + 3\right) + \left(A_{41}\cos L + B_{41}\sin L\right)$$
$$\times \frac{5}{2}\cos\theta\left(7\sin^3\theta - 3\sin\theta\right) + \dots\dots$$

also adds to the ellipticity of the geoid. It is assumed that such terms are of $0\,(\epsilon^2)$.

The coefficients A_{22}, B_{22} define the ellipticity η of the equator. Here again, there will be terms in Y_4, Y_6 etc. which contribute to this ellipticity, but as mentioned in chapter II, η has so far been determined by considering only the A_{22} and B_{22} terms. This equatorial ellipticity is very small and has not been determined reliably.

We will next consider the significance of the harmonic terms $\cos L\cos 2\theta$, $\sin L\sin 2\theta$. The expression $\frac{3}{2}\,(A_{21}\cos L + B_{21}\sin L)\times \sin 2\theta$ does not contribute to either the meridional or equatorial ellipticity since it vanishes both for $\theta = 0$ and $\theta = 90°$. The equation

$$r = k\left\{1 - \frac{2}{3}\epsilon P_2 + \frac{1}{2}\eta\cos^2\theta\cos 2L \right.$$
$$\left. + \frac{3}{2}\left(A_{21}\cos L + B_{21}\sin L\right)\sin 2\theta\right\} \quad \dots \quad (6\cdot6)$$

represents an ellipsoid referred to axes which are not the principal axes. To see this, consider for simplicity the spheroid obtained by putting $\eta = 0$ in the above. Let its principal axes be OC_0, Ox, Oy, (Fig. 13) where the plane C_0Ox passes through Greenwich. If S is any point on the spheroid such that $\angle C_0OS = 90 - \theta$, its equation referred to these axes is

$$r = a\left(1 - \epsilon\sin^2\theta\right) = k\left(1 - \frac{2}{3}\epsilon P_2\right) \quad \dots \quad (6\cdot7)$$

Now choose a new set of rectangular axes OC, Ox' Oy', where OC is defined with respect to OC_0 (the minor axis) by the angular coordinates (θ_0, L_0), L_0 being reckoned from the plane through C_0 and Greenwich. We have

$$\sin\theta' = \sin\theta\sin\theta_0 + \cos\theta\cos\theta_0\cos(L - L_0)$$
$$\doteqdot \sin\theta + \left(\frac{\pi}{2} - \theta_0\right)\cos\theta\cos\left(L - L_0\right), \text{ since } \theta_0 \doteqdot \frac{\pi}{2}.$$

The equation of the spheroid referred to the new axes is

$$
\begin{aligned}
r &= a\,(1-\epsilon\sin^2\theta')\\
&= a\left\{1-\epsilon\sin^2\theta-\epsilon\left(\frac{\pi}{2}-\theta_0\right)\sin 2\theta\cos\left(L-L_0\right)\right\}\\
&= k\left\{1-\frac{2}{3}\epsilon P_2-\epsilon\left(\frac{\pi}{2}-\theta_0\right)\sin 2\theta\cos\left(L-L_0\right)\right\} \qquad (6\cdot 8)
\end{aligned}
$$

Comparing this with equation (6 · 6), we have

$$
\left.\begin{aligned}
\frac{3}{2}A_{21} &= -\epsilon\left(\frac{\pi}{2}-\theta_0\right)\cos L_0\\
\frac{3}{2}B_{21} &= -\epsilon\left(\frac{\pi}{2}-\theta_0\right)\sin L_0
\end{aligned}\right\} \quad \ldots \quad (6\cdot 9)
$$

Hence

$$\cot L_0 = \frac{A_{21}}{B_{21}}, \quad \ldots \quad (6\cdot 10)$$

$$\frac{\pi}{2}-\theta_0 = \frac{1}{\epsilon}\cdot\frac{3}{2}\sqrt{A_{21}^2+B_{21}^2}. \quad \ldots \quad (6\cdot 11)$$

The above shows that the coefficients of the harmonics $\cos L \sin 2\theta$ and $\sin L \sin 2\theta$ depend on the deviation of the rotation axis from the axis of symmetry*.

In the case of the earth, θ_0, L_0 can be deduced by utilizing Prey's† results. He has expressed the lithosphere in a series of spherical harmonics up to terms of the 16th order. His series A gives the undulations of the lithosphere from mean sea-level, and series B is such that over the oceans it gives the same results as A, but on land it gives zero values. The values of the coefficients of the various terms in the two series are tabulated in Table VII of the above publication by Prey.

Substituting the values A_{21} and B_{21} in equations (6 · 10) and (6 · 11), we obtain

$$\frac{\pi}{2}-\theta_0 \doteqdot 1^\circ$$

and

$$L_0 \doteqdot 130^\circ \text{ W}.$$

This displacement of 1° of the axis of rotation from the minor axis is excessive. If topography were compensated to a depth of 100 km., Mader has estimated that this displacement would be of the order of 1′ of arc. Even a displacement of 1′ is excessive because results of the variation of latitude point to a coincidence of the two axes to within a fraction of a second of arc. Hence there must be some sort of compensation which must be neutralizing the above displacement.

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* Lambert, Bull. Geod., No. 26, 1930, 112.

† Abhandlungen der Königlichen Gesellachaft der Wissenschaften zu Göttingen, Neue Folge Bd. xı, 1, Berlin 1922.

3. Definitions of a reference surface.—We have already seen in chap. I how Helmert* used the level spheroid as a reference surface to the geoid. The potential on the geoid is

$$W=\frac{fM}{r}\left\{1-\frac{3K}{2r^2}\left(\sin^2\theta-\frac{1}{3}\right)+\frac{3}{4}\frac{B-A}{Mr^2}\cos^2\theta\cos 2L\right\}$$
$$+\frac{fY_3}{r^4}+\frac{fY_4}{r^5}+\ldots\ldots+\frac{1}{2}\omega^2 r^2\cos^2\theta$$

and on the level spheroid is $U=W_0$, where

$$U=\frac{fM}{r}\left\{1+\frac{K}{2r^2}(1-3\sin^2\theta)+\frac{\omega^2 r^3}{2fM}\cos^2\theta\right.$$
$$\left.+\frac{D}{r^4}\left(\sin^4\theta-\frac{6}{7}\sin^2\theta+\frac{3}{35}\right)\right\}.$$

By equation (1·28), the formula for normal gravity on the above level spheroid is

$$g=G_e\left\{1+\left(\frac{5}{2}m-\epsilon+6\epsilon^2-\frac{1}{2}\epsilon m-\frac{19}{7}\delta\right)\sin^2\theta-(7\epsilon^2-3\delta)\sin^4\theta\right\}.$$

This formula serves as a basis from which to reckon gravity anomalies.

The following is a more illustrative treatment for showing the analytical relations between the geoid and its reference surface in gravity work. For the formulæ of the preceding chapter (giving the undulations of the geoid from the gravity anomalies) to be valid, the reference surface has to satisfy certain conditions. Choosing the origin at the centre of gravity of the geoid, its equation may be written as

$$r=k(1+Y_2+Y_3+\ldots\ldots).$$

We have seen that with the reference surface

$$r=k(1-\tfrac{2}{3}\epsilon P_2),$$

equations (5·6) are accurate to the small quantities of order ϵ^2 provided G is the same for both the surfaces. Since $G=\frac{fY_0}{k}-\frac{2}{3}\omega^2 k$, this condition ensures that the masses of the two surfaces are equal. Obviously the centres of gravity of the two surfaces are also identical.

To obtain a general result, suppose the equation of the geoid is written in the form

$$r=k(1-\epsilon\Sigma Y_n-\epsilon_1\Sigma Z_n). \qquad \ldots \quad (6\cdot 12)$$

Then by chap. V, para 3, Stokes' equations (5·6) are satisfied correct to terms of $O(\epsilon^2)$ if the reference surface be chosen as

$$r=k(1-\epsilon\Sigma Y_n).$$

By our definition of the ellipticity of the geoid, this choice ensures that the mean meridional ellipticity is the same for both the surfaces, the difference $d\epsilon$ in their ellipticities being of $O(\epsilon_1)=O(\epsilon^2)$.

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* Höheren Geodäsie, **2**, 1884, 89.

If, then, the geoid is

$$r=k\left\{1-\frac{2}{3}\epsilon P_2+\epsilon'\Sigma v_n\right\}, \qquad \ldots(6\cdot 13)$$

Stokes' equations would be satisfied if the reference surface be chosen as any surface of the family

$$r=k\,(1-\tfrac{2}{3}\epsilon P_2+\epsilon'\Sigma w_n), \qquad \ldots(6\cdot 14)$$

provided ϵ' is of $0\,(\epsilon^2)$. As an example, let the geoid be

$$r=a\left\{1-\epsilon\sin^2\theta-\frac{3}{2}\epsilon^2\sin^2\theta\cos^2\theta+\chi\sin^2\theta\cos^2\theta\right\} \quad \ldots(6\cdot 15)$$

In terms of spherical harmonics, it can be written as

$$r=k\left[1-\frac{2}{3}\epsilon P_2+P_2\left(\frac{2}{21}\chi-\frac{23}{63}\epsilon^2\right)-P_4\left(\frac{8}{35}\chi-\frac{12}{35}\epsilon^2\right)\right]. \quad (6\cdot 16)$$

Comparing with $(6\cdot 13)$, we have

$$\epsilon'\sum_{n=2} v_n=P_2\left(\frac{2}{21}\chi-\frac{23}{63}\epsilon^2\right)-P_4\left(\frac{8}{35}\chi-\frac{12}{35}\epsilon^2\right).$$

If we choose as our reference surface the spheroid

$$r=a\left(1-\epsilon\sin^2\theta-\frac{3}{2}\epsilon^2\sin^2\theta\cos^2\theta\right), \qquad \ldots(6\cdot 17)$$

we have $$\epsilon'\sum w_n=-\frac{23}{63}\epsilon^2P_2+\frac{12}{35}\epsilon^2P_4.$$

Gravity on $(6\cdot 16)$ is

$$g=G\left[1+\left(\frac{5m-2\epsilon}{3}+\frac{64}{63}m\epsilon-\frac{5}{9}\epsilon^2-\frac{25}{18}m^2+\frac{2}{21}\chi\right)P_2\right.$$
$$\left.-\left(\frac{12}{7}m\epsilon-\frac{4}{5}\epsilon^2+\frac{24}{35}\chi\right)P_4\right]$$

and on $(6\cdot 17)$ is

$$\gamma_0=G\left[1+\left(\frac{5m-2\epsilon}{3}+\frac{64}{63}\epsilon m-\frac{5}{9}\epsilon^2-\frac{25}{18}m^2\right)P_2-\left(\frac{12}{7}m\epsilon-\frac{4}{5}\epsilon^2\right)P_4\right],$$

the mean value G of gravity being the same on both the surfaces (*cf.* chap. I, para 6).

From these equations, we see that

$$\left.\begin{aligned}N&=k\left(\frac{2}{21}\chi P_2-\frac{8}{35}\chi P_4\right)\\ \Delta g&=G\left(\frac{2}{21}\chi P_2-\frac{24}{35}\chi P_4\right)\end{aligned}\right\}$$

This confirms that Stokes' equations are valid for our reference surface.

In view of what has been said in para 2, it is obvious that the reference surface has the same volume as the geoid. Assuming the same mean density for the two surfaces, their masses must also be identical. When the geoid is reckoned as the equipotential of a reference surface, having a coating of total mass zero on it, the condition which the reference surface has to satisfy in order that Stokes' equations should hold is that it should have the same potential as the geoid (*cf.* chap. v, para 3). It appears, then, that to the order of accuracy to which we are working, the reference surface is so deformed by the superposition of coating on it, that the new level surface having the same potential will also have its volume equal to it. This convenient property only holds when the mass displacements are small. In particular, it is true for the important case of a massless coating on a sphere. It implies that the N's found by Stokes' method satisfy the condition

$$\iint N\, d\omega = 0.$$

It is important to realize the significance of the above condition which the reference surface has to satisfy. Michailov* has tested the accuracy of Stokes' quadrature formula by numerical examples with simple models. He starts with the International spheroid and finds its separation from a reference sphere of equal volume. The result by Stokes' formula is out by 26 metres which is of $0\,(a\epsilon^2) = 70$ metres (and at first sight this shows that our formula is accurate to $0\,(\epsilon)$ only). Michailov appears to be satisfied that Stokes' formula has given N to this accuracy. Actually, however, it gives N to $0\,(a\epsilon^3)$ i.e., $\frac{1}{4}$ metre or so, provided the reference surface is properly chosen. Michailov's reference sphere does not satisfy the condition that it has the same mean meridional ellipticity as the geoid.

Since $\Delta g = G\Sigma\,(n-1)\,u_n$, we have

$$\iint \Delta g\, d\omega = 0. \qquad \ldots (6\cdot 18)$$

The area of the zone $\psi_k - \psi_{k+1}$ is $2\pi\, k^2\,(\cos\psi_k - \cos\psi_{k+1})$. If $\Delta_m g$ donotes the mean value of gravity in this zone, equation $(6\cdot 18)$ may be written as

$$\Sigma\, 2\pi\, k^2\, \Delta_m g\;(\cos\psi_k - \cos\psi_{k+1}) = 0.$$

Hence the reference surface also satisfies the condition that the mean value of gravity on it is the same as on the geoid.

4. Relation between the centres of gravity of the geoid and its reference spheroid†.—A reference spheroid is the equipotential of certain masses within it. We have seen that these masses cannot be homogeneous. The geoid is the equipotential of the spheroid with a coating of total mass zero superposed on it. Both these surfaces being heterogeneous, the centres of gravity of mass and volume are not identical. It is important to realize, however, that for a nearly spherical equipotential surface, the masses have to be so arranged that the two centres of gravity are coincident. To see this, let O_m, O_v denote the centres of gravity of mass and volume respectively. We have already seen that if

* Verh. der 8-ten Tagung der Balt. Geodät. Komission, Helsinki 1936, 207.

† i.e., Centres of gravity of the matter contained within these two surfaces.

the origin be chosen at O_m, the expression for the potential at an external point is given by (1·24). Next, with respect to O_e as origin, the equation of the nearly spherical level surface may be written as

$$r = k\,(1 + u_2 + u_3 + \cdots\cdots),$$

and we have seen in chap. I, para 6 that its external potential is

$$W = fY_0\left(\frac{1}{r} + \frac{k^2 u_2}{r^3} + \cdots\cdot\right) - \frac{\omega^2 k^5}{2r^3}\left(\frac{1}{3} - \sin^2\theta\right) \quad \ldots(6\cdot 19)$$

Equations (1·24) and (6·19) are identical; hence the two centres of gravity are coincident. It is worth while calling attention to the fact that this property only holds for a level surface which has no masses external to it. In other words, it holds for the compensated geoid but not for the natural geoid. In deriving the quadrature formula between N and Δg, the equations of the geoid and its reference spheroid are so chosen that the term u_1 is absent from both. In other words, the centres of gravity of the masses inside them (or of their volumes) are identical.

Suppose, however, we want to find g on the level surface

$$r = k\,(1 + u_1 + u_2 + \cdots\cdot) \quad \ldots \quad (6\cdot 20)$$

It is not difficult to see that the u_1 term will be missing from the expression for g. Thus, let the potential of the attracting matter be

$$V = \frac{Y_0}{r} + \frac{Y_1}{r^2} + \cdots\cdots$$

If (6·20) is to be an equipotential, we must have

$$W = V + \tfrac{1}{2}\,\omega^2 r^2 \cos^2\theta = \text{constant on it},$$

or

$$W = f\,Y_0\left(\frac{1}{r} + \frac{k u_1}{r^2} + \frac{k^2 u_2}{r^3} + \cdots\cdot\right) - \frac{\omega^2 k^5}{2r^3}\left(\frac{1}{3} - \sin^2\theta\right),$$

from which we get

$$g = -\left(\frac{\delta W}{\delta r}\right)_{r=k}$$

$$= \frac{fY_0}{k^2}\left[1 - 2u_0 + u_2 + 2u_3 + \cdots\cdot - \frac{5}{2}\,\omega^2 k\left(\frac{1}{3} - \sin^2\theta.\right)\right]$$

The term in u_1 disappears in the final expression for g. In other words, the values of gravity at corresponding points on the two surfaces

$$r = a\,(1 + Y_0 + Y_1 + Y_2 + \cdots\cdot)$$

$$\text{and } r = a\,(1 + Y_0 + Y_2 + \cdots\cdots\cdots)$$

are identical.

The same holds for the gravity anomaly Δg which is the difference of gravity on the two surfaces. To see this, consider a sphere $r = a$ and call it surface I. Put a coating $a\,(Y_0 + Y_1 + Y_2 + \cdots\cdots\cdot)$ on it, and we get a new level surface II. The potential due to this coating is

$$S = 4\pi f\,a\,\sigma\,\Sigma\,\frac{1}{2n+1}\left(\frac{a}{r}\right)^{n+1} Y_n.$$

If g denotes gravity on II, and γ_0 on I, we have

$$g-\gamma_0 \doteqdot -\left(\frac{\delta S}{\delta r}+\frac{2S}{r}\right)$$

$$=4\pi f\sigma\left[Y_0+\frac{2}{3}Y_1+\frac{3}{5}Y_2+\cdots\right]-8\pi f\sigma\left[Y_0+\frac{1}{3}Y_1+\frac{1}{5}Y_2+\cdots\right]$$

As before, the term in Y_1 becomes zero automatically. If, then, we are given that $\Delta g=A_1\,u_1+A_2\,u_2+\cdots\cdots$ and we want to determine $N=B_1\,u_1+B_2\,u_2+\cdots\cdots$, we see that all the terms are determinable except B_1. Other considerations are needed for fixing this term. The above property depends on the law of attraction and is independent of the internal constitution of the body.

In the quadrature formula $N=\frac{k}{2\pi G}\iint \Delta g\,f(\psi)\,d\omega$, a term of the type $A_0+A_1u_1$ in Δg has no effect on N. Hence for deducing the undulations of the geoid from Δg's, the position of the centre of the reference surface has to be defined beforehand; it cannot be derived from gravity observations. The simplest course is to make the two centres of gravity coincide. In the converse case, when we know the orientation of the reference surface and its separation $N=B_1\,u_1+B_2\,u_2+\ldots\ldots$ from the geoid, Δg can be easily determined except that one constant G or M has to be found by some other method. Although N contains the u_1 term, we have seen that this term will be missing from Δg. Hence the physical and dynamical definition of the reference surface of the geoid, which involves the idea of the equality of the potential, ensures the coincidence of the centres of gravity of the two surfaces.

To carry the discussion a bit further, suppose we choose the centre of gravity of volume of the earth as origin. Its equation may be written as $r=k\,(1+Y_2+Y_3+\ldots+Y_n+\ldots)$. ... (6·21)
Regarding this as a sphere $r=k$ with mean density ρ_m, on which a coating of surface density $k\rho\sum_{n=2} Y_n$ has been superposed, we see that the equation of the level surface which has the same potential as the sphere $r=k$ is

$$r=k+k\frac{3\rho}{\rho_m}\left(\frac{Y_2}{5}+\frac{Y_3}{7}+\ldots\ldots\right). \qquad \ldots \quad (6\cdot 22)$$

The first harmonic term being absent, the centre of gravity of this level surface coincides with that of the earth. This coincidence must be to first order terms only, because we have seen in para 2 that strictly speaking, terms Y_2^2, etc. enter in the expression for the centre of gravity of a surface. From the foregoing discussion we see that if an uncompensated coating $k\sum_{n=2} Y_n$ is superposed on a sphere $r=k$, which is a level surface of certain attracting masses within it, the new level surface with the same potential will have the same centre of gravity as the sphere.

We will now consider the relations between the centres of gravity of the earth, the natural geoid and the compensated geoid. Let E be the centre of gravity of mass of the earth, G of the geoid, and C that of the compensated geoid (Fig. 14). Laborious computations are needed for obtaining the numerical estimates of the relative distances between these points.

Prey's series for the lithosphere gives

$a\,A_{11} = 1129 \cdot 8^{\text{metres}}, \quad a\,B_{11} = 664 \cdot 4^{\text{metres}}, \quad a\,A_1 = 1263 \cdot 8^{\text{metres}}.$

and for the hydrosphere

$a\,A_{11} = 1005 \cdot 5^{\text{metres}}, \quad a\,B_{11} = 563 \cdot 8^{\text{metres}}, \quad a\,A_1 = 1112 \cdot 3^{\text{metres}},$

where A_{11}, B_{11}, A_1 are constant coefficients in the expression for Y_1. Based on the above data, Lambert* has tabulated the displacements of the centres of gravity due to superposition of continents and oceans. If $(\bar{r}, \bar{\phi}, \bar{L})$ denote the amount and direction of the displacement of the centre of gravity, then for no compensation we have

Lambert—Prey	Mader
$\bar{r} = 625^{\text{metres}}$	672^{metres}
$\bar{\phi} = 43^\circ\ 57'$ N.	$49^\circ \cdot 6$ N.
$\bar{L} = 31^\circ\ 01'$ E.	$34^\circ \cdot 2$ E.

and for compensation at the depth of 100 km.

$\bar{r} = 4 \cdot 9^{\text{metres}}$	$5 \cdot 8^{\text{metres}}$
$\bar{\phi} = 43^\circ\ 57'$ N.	$49^\circ \cdot 2$ N.
$\bar{L} = 31^\circ\ 01'$ E.	$34^\circ \cdot 1$ E.

It is important to notice that by isostatic compensation the displacement of the centre of gravity is reduced to 15 feet. In our figure, therefore, $GC = 15$ feet. Of course if we assume some different type of compensation we will get a different answer. Indeed by assuming suitable subterranean mass anomalies, we can make the difference of 625 metres in the centres of gravity of the normal and final earth to disappear.

When the mass displacement is such that the centre of gravity is displaced by an appreciable amount, Stokes' formula would still hold, provided the new level surface be shifted so that the centre of gravity of the new mass configuration is made to coincide with that of the original level surface.

5. Mean load level.—W. D. Lambert† has introduced yet another reference surface, which he designates as the mean load level. Imagine the oceans of the earth to be solidified into matter of normal crustal density. Take a *spheroidal equipotential* having the same volume as the modified earth. Assuming the earth's surface to be 70·8% ocean and 29·2% land, and taking the mean depth of the oceans to be 3800 metres and that of the land to be 840 metres, the mean load level surface comes out to be about 4600 feet below the geoid.

* Bull. Geod., No. 26, 1930, 111.
† Bull. Geod., No. 26, 1930, 21.

If the topography be reckoned from this surface, then under certain conditions, the deformation of the geoid due to the introduction of topography and its compensation is given by the simple formula

$$u = \frac{3}{4} \frac{\rho}{\rho_m} \cdot \frac{\tau}{k} H,$$

where H denotes the height of the topography reckoned above the mean load level, τ the depth of compensation and ρ, ρ_m the crustal and mean densities of the earth respectively.

Darling* has tested the accuracy of this approximate formula by considering 31 stations, of which 22 are on land and 9 in the sea, located in the Atlantic, Pacific and the Arctic oceans and also in the waters of the E. Indies. The average discrepancy from the true value came out to be 5 feet and the greatest discrepancy 12 feet.

It should be mentioned, however, that now we can get u more precisely with the help of Lambert's† tables which are based on more rigorous formulæ.

6. Earth spheroid and reference spheroid.—It is important to realize the difference between spheroids used in gravity work and those used for computing triangulation and deflections. A gravity spheroid is unique and may be termed the 'Earth spheroid.' As we have seen, it has the same centre of gravity and mass as the matter within the geoid. The reference surface in triangulation has to be a true spheroid, which may be defined in two alternative ways by seven constants as below:

(a) (x_0, y_0, z_0), the co-ordinates of its centre.

(β, γ), the direction cosines of its minor axis.

(a, ϵ), its semi-major axis and ellipticity.

(b) ζ_0, the angle between the spherical and geoidal normals at an arbitrarily chosen point, known as the geodetic datum.

A_0, the angle which the plane containing the above two normals makes with the geoidal meridian

N_0, the vertical separation between the spheroid and the geoid at the datum.

$(\beta, \gamma, a, \epsilon)$ as before.

It is easy to show that the quantities (ζ_0, A_0, N_0) fix the co-ordinates of the centre of the spheroid uniquely. In triangulation, the centre (x_0, y_0, z_0) is not defined to be at the earth's centre of gravity. It is defined by assigning arbitrary values to ζ_0, A_0, N_0. The angular co-ordinates β, γ are specified by defining the minor axis of the spheroid to be parallel to the earth's axis of rotation. This condition enables Laplace's equation to be utilized.

* Bull Geod., No. 44, 1934.

† U. S. Coast and Geodetic Survey, Sp. Publication No. 199.

In determining the figure of the earth by triangulation in different countries, one is handicapped by the fact that the datums are unconnected. Each triangulation is computed on a differently orientated spheroid, which is unsatisfactory. There is no immediate prospect of connecting the different triangulations of the globe, as the oceans present a serious difficulty. The best that one can do is to derive the dimensions of the best fitting spheroid from each isolated triangulated region, and combine the various results by assigning suitable weights.

Hence it is much more practicable to connect gravity data of different countries rather than their triangulations. Apart from this, the triangulations of different countries are on different spheroids, and the problem of conversion of a triangulation series from one spheroid to another is much more complicated than that of conversion of a gravity formula. The determination of the figure of the earth from the gravity anomalies, therefore, possesses a more absolute character. When, however, enough gravity data are available on the globe, it will be possible to place each astronomic-geodetic net on its reference spheroid in terms of the Earth spheroid. For each isolated triangulation net, if at one point (N, η, ξ) are determined by Stokes' theorem, the net can be computed in terms of the reference spheroid with the origin at the earth's centre of gravity.

In India, the International spheroid is orientated by making it fit the compensated geoid best. Due to dearth of gravity data, it is not possible to get reliable values of (N, η, ξ) at the datum from Δg's by the formulæ of chapter v. If Hirvonen's results could be accepted, one could at least determine its separation from the geoid at Kaliānpur in *International terms*. Hirvonen's results seem to indicate that the International spheroid, as orientated in India, has to be depressed through 200 feet or so.

7. Summary.—In this chapter the various types of reference surfaces are defined. It is pointed out that it is necessary to distinguish a triangulation reference spheroid from a gravity one, even if their dimensions are identical. The geoidal profiles ordinarily determined from gravity data are not closely linked with those from deflection data, as the orientation of the reference spheroid may be quite different in the two cases. They can only be anchored to each other satisfactorily, provided (N, η, ξ) are derived at one point of the triangulation net by the help of Stokes' theorem.

G.B.—P.O.—J.S. 243—21-12-39—375 books.

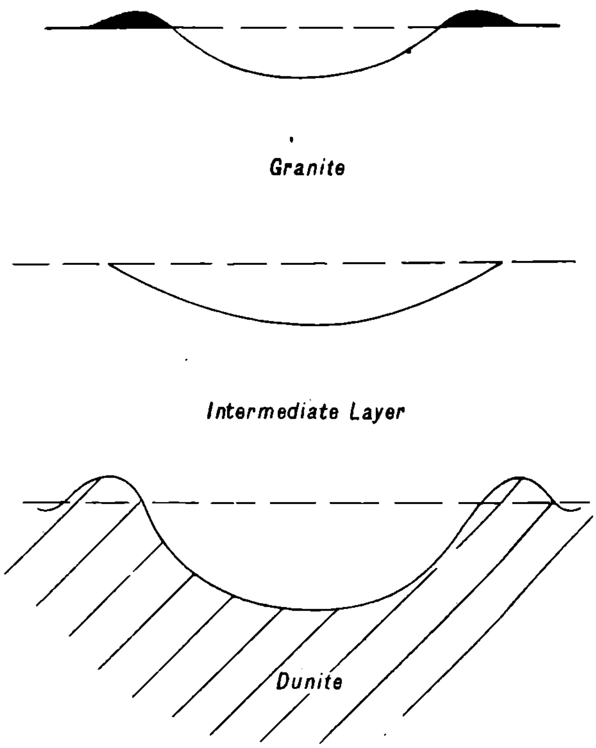

Fig. 1

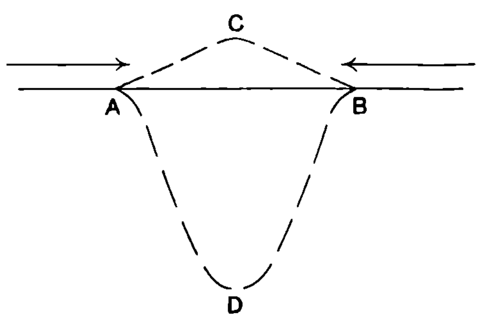

Fig. 2

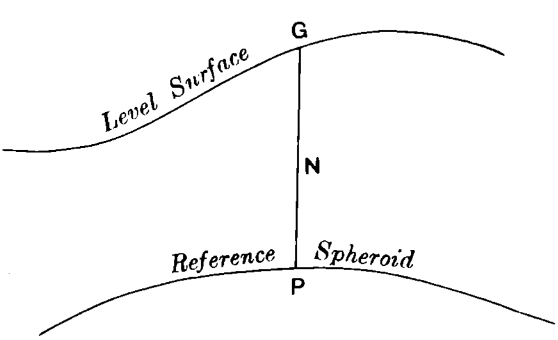

Fig. 3

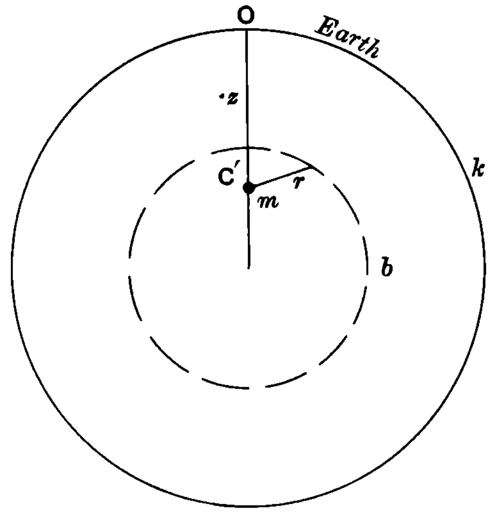

Fig. 4

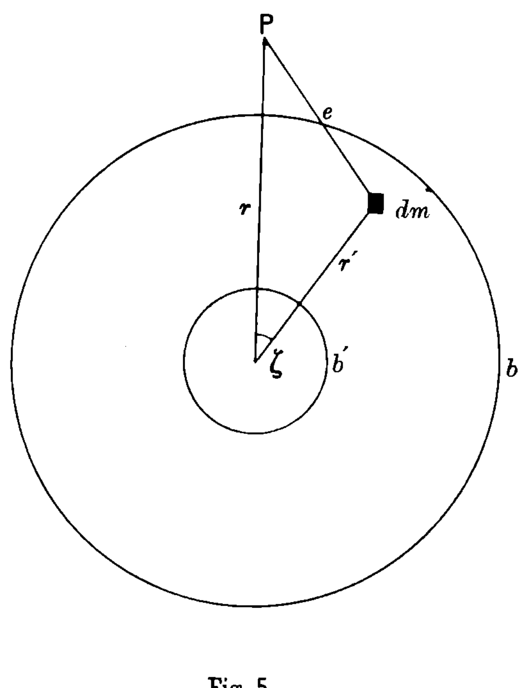

Fig. 5

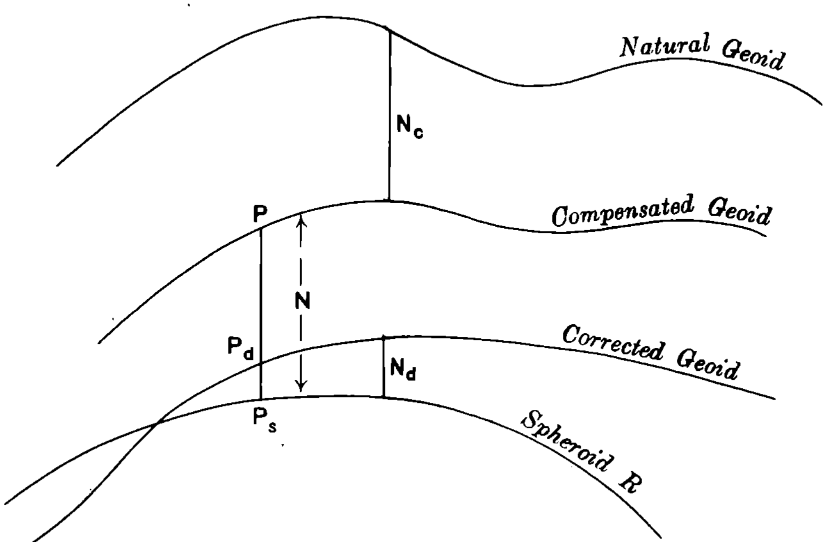

Fig. 6

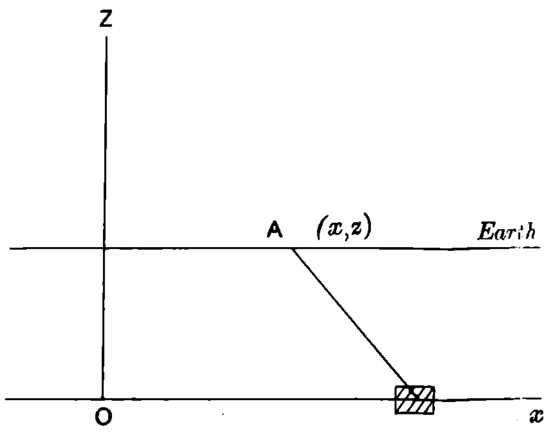

Fig. 7

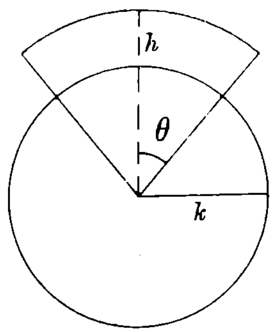

Fig. 8

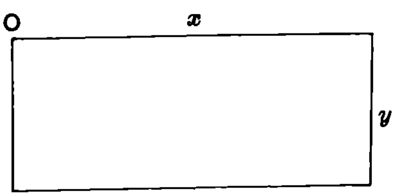

Fig. 9

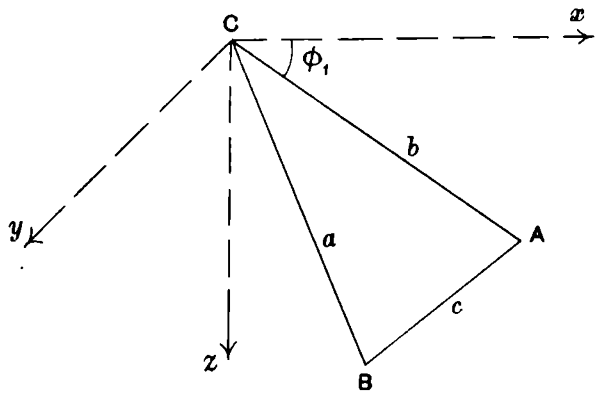

Fig. 10

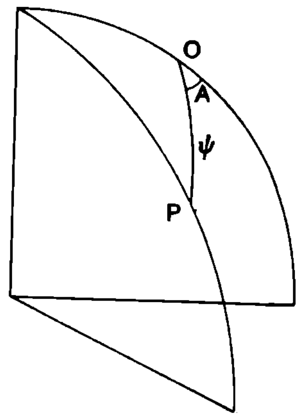

Fig. 11

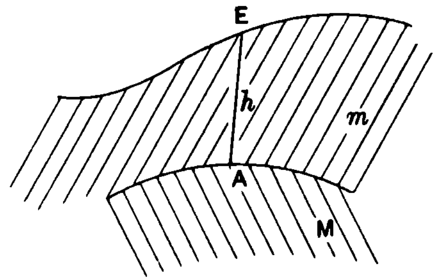

Fig. 12

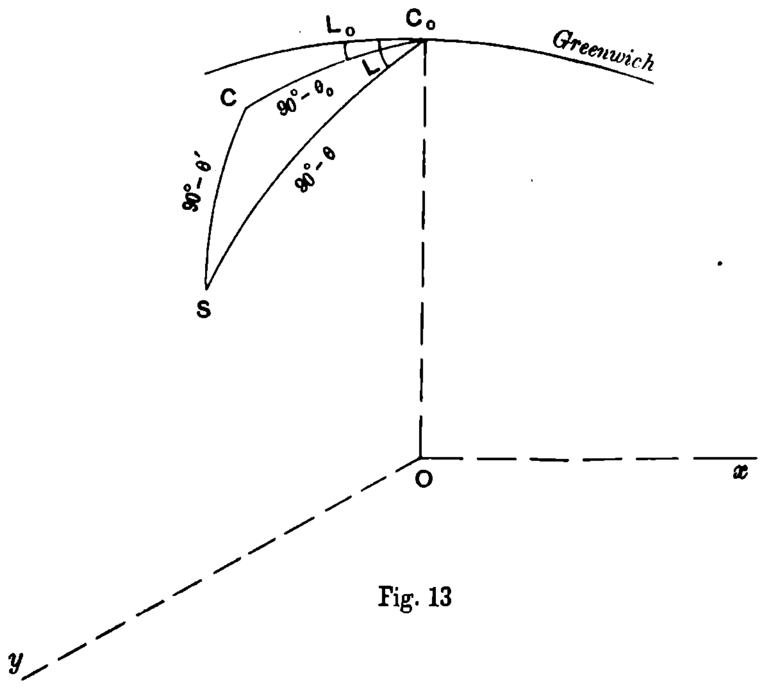

Fig. 13

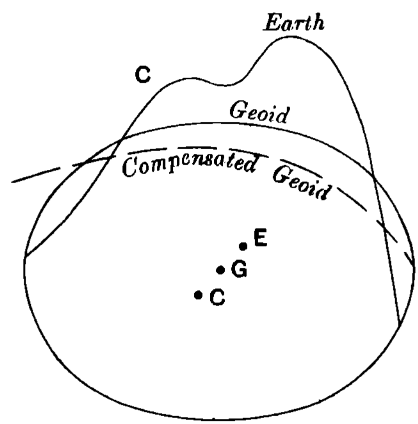

Fig. 14